APPM 3310: Matrix Methods — Final — May 2, 2009

On the front of your bluebook print (1) your name, (2) your student ID number, and (3) a grading table. **Explain all of your answers.** This test is worth 150 points. There are 20 additional points available if you choose to answer all the questions. A correct answer with no supporting work may receive no credit while an incorrect answer with some correct work may receive partial credit. No books or notes. No electronic devices of any kind (e.g. cell phones, calculators, etc.) are permitted. Begin each problem on a new page.

1. (30 points) Let $\mathbb{P}^{(4)}$ denote the vector space of all polynomials of degree less than or equal to 4.
   (a) Are $p_1(x) = x-2$, $p_2(x) = x^2-5x+4$, $p_3(x) = 3x^2-4x$, $p_4(x) = x^2-1$ linearly independent elements of $\mathbb{P}^{(4)}$?
   (b) What is the dimension of $V = \text{span}\{p_1, p_2, p_3, p_4\}$?
   (c) Verify the *Cauchy-Schwartz* inequality for the functions $p_1 = x - 2$ and $p_4 = x^2 - 1$ with respect to the L^2-norm on [0, 1].

2. (40 points) Let $K = K^T$ be a symmetric $n \times n$ matrix.
   (a) We learned that K is positive definite if and only if all of its eigenvalues are strictly positive. Prove the direction: $K > 0$ implies all of its eigenvalues are strictly positive.
   (b) Prove that K^2 is positive definite.
   (c) Prove that if K is positive definite then K can be written as a Gram matrix.
   (d) Find the Gram matrix K for the monomials 1, x, x^2 under the L^2 inner product on [0,1].

3. (50 points) For this problem let
$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$
   (a) Find the eigenvalues and eigenvectors of A.
   (b) Is A positive definite, positive semi-definite or neither? Explain.
   (c) Find orthonormal eigenvector bases for each of the four fundamental subspaces of A.
   (d) Find and orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of A. Verify orthonormality for full credit.
   (e) Write out the spectral factorization of A.

4. (40 points) A few unrelated short answer questions.
   (a) State the Fundamental Theorem of Linear Algebra parts 1 and 2. (Hint: part 1 characterizes the dimensions of the four fundamental subspaces of a matris; part 2 gives orthogonality relations between them.)
   (b) Give the definition for W to be a subspace of a vector space V.
   (c) State how positive definite matrices are related to inner products on $\mathbb{R}^n$.
   (d) Suppose the vector space $V = span\{\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_n\}$. What do you know about $dim(V)$?

5. (10 points) For the quadratic form $q(\mathbf{x}) = 2x_1^2 + x_1x_2 - 2x_1x_3 + 2x_2^2 - 2x_2x_3 + 2x_3^2$, find the vector $\mathbf{v} \in \mathbb{R}^3$ that minimizes $q(\mathbf{x})$. (Hint: use a matrix to represent $q(\mathbf{x})$.)