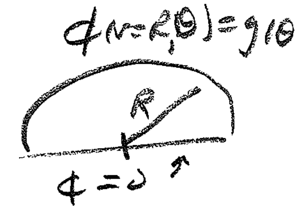
Ex. II   Dec 9, 2009

1. $\nabla^2 \phi = 0$



$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

Sep. Var. $\phi_s = F(r)\,G(\theta)$

$$\frac{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right)}{F} + \frac{1}{r^2}\underbrace{\frac{\frac{\partial^2 G}{\partial \theta^2}}{G}}_{=-n^2} = 0$$

$\Rightarrow$

$G_{\theta\theta} + n^2 G = 0 \qquad G = c_{1n}\cos n\theta + c_{2n}\sin n\theta$

$r\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) - n^2 F = 0 \qquad F = r^p \Rightarrow$

$r^2 F_{rr} + r F_r - n^2 F = 0 \quad \Rightarrow p^2 - n^2 = 0$

$p = \pm n \quad n \neq 0$

$F = \alpha_0 + \alpha_1 \log r$
$n = 0$

$$\phi_s = \begin{cases} \cancel{r^n}(A_n\cos n\theta + B_n\sin n\theta) & \text{not bdd.} \\ r^{-n}(A_n\cos n\theta + B_n\sin n\theta) \end{cases}$$

$$\phi = A_0 + \sum_{n=1}^{\infty} r^{-n}(A_n\cos n\theta + B_n\sin n\theta)$$

$$\left.\begin{aligned} \phi = 0 &= A_0 + \sum r^{-n} A_n \\ 0 &= A_0 + \sum r^{-n} A_n(-)^n \end{aligned}\right] \text{all } r:\ 0 < r < 1 \Rightarrow A = A_n = 0$$

$\phi(r,\theta) = \sum_{n=1}^{\infty} B_n r^{-n}\sin n\theta \qquad r = R:\ g(\theta) = \sum_{n=1}^{\infty} B_n R^{-n}\sin n\theta$

$$\Rightarrow \boxed{B_n = \left(\int_0^\pi g(\theta)\sin n\theta\, d\theta\right)\frac{R^n}{\int_0^\pi \sin^2 n\theta\, d\theta}} \qquad \int_0^\pi \sin^2 n\theta\, d\theta = \pi/2$$

if $g(\theta) = g_0 \sin\theta \qquad B_n = 0\ n \neq 1 \qquad B_1 = g_0 R.$

2. $\frac{\partial T}{\partial t} = \kappa \nabla^2 T$, $\kappa > 0$: $\frac{\partial T}{\partial r}(r=R,\theta,t) = 0$

$T(r,\theta,t=0) = f(r)$

$T(r,\theta,t) = T(r,t)$ only depends on r : $0<r<R$

$T(r,t) = f(r)G(t)$

$$\frac{G_t}{\kappa G} = \frac{\frac{1}{r}\left(\frac{\partial}{\partial r} r \frac{\partial f}{\partial r}\right)}{f} = -\lambda$$

$G_t = -\lambda \kappa G$ $\quad G = G_0 e^{-\lambda \kappa t}$

$\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) + \lambda r f = 0 \Rightarrow r^2 \frac{\partial^2 f}{\partial r^2} + r \frac{\partial f}{\partial r} + \lambda r^2 f = 0$

not Bdd

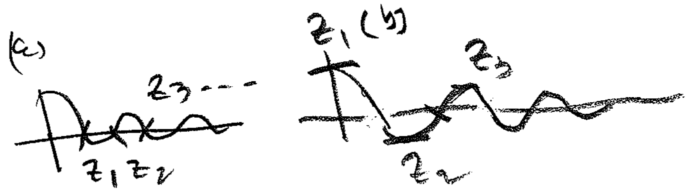
$f = C_1 J_0(\sqrt{\lambda} r) + \cancel{C_2 Y_0(\sqrt{\lambda} r)}$ $\quad$ (a) $J_0(z_n) = 0$

$f_r = C_1 \sqrt{\lambda} J_0'(\sqrt{\lambda} R) = 0$ $\quad$ (b) $J_0'(z_n) = 0$

i.e. max/min of $J_0(z)$

$\lambda_n = (z_n/R)^2$

or zeros of $J_0'(z)$



$$T(r,t) = \sum_{n=1}^{\infty} A_n e^{-\kappa \left(\frac{z_n}{R}\right)^2 t} J_0(\sqrt{\lambda_n} r) \rightarrow \begin{matrix} 0 & \text{case (a)} \\ A_0 & \text{case (b)} \\ (z_1 = 0) & \end{matrix}$$

$\int_0^R J_0(\sqrt{\lambda_n} r) J_0(\sqrt{\lambda_{n'}} r) r\, dr = 0 \quad n \neq n'$

$t=0$: $\quad f(r) = \sum_{n=1}^{\infty} A_n J_0(\sqrt{\lambda_n} r)$

$$A_n = \frac{\int_0^R f(r)\, r\, J_0(\sqrt{\lambda_n} r)\, dr}{\int_0^R r\, J_0^2(\sqrt{\lambda_n} r)\, dr}$$

3. (a) $\phi_{xx} + \lambda\phi = 0 \qquad \phi(0) = 0 \qquad \phi_x(L) = 0$

$\phi(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$

$\phi(0) = 0 = c_1 \qquad \phi(x) = c_2 \sin\sqrt{\lambda}x$

$\phi_x = c_2\sqrt{\lambda}\cos\sqrt{\lambda}L = 0 \qquad \sqrt{\lambda_n}L = (n+\frac{1}{2})\pi$

$\lambda_n = \left((n+\frac{1}{2})\frac{\pi}{L}\right)^2 > 0 \quad n = 0, 1, 2, \dots$

$\phi_n = \sin(n+\frac{1}{2})\frac{\pi x}{L}$; RQ $\Rightarrow$ $\underline{\lambda_n > 0}$

(b) (i) $\lambda > 0$ $\quad \phi(x) = c_2 \sin\sqrt{\lambda}x$

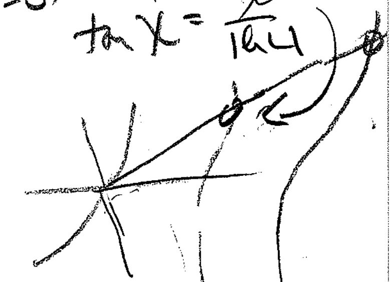
$(\phi_x + h\phi)(L) = 0 \qquad \sqrt{\lambda}\cos\sqrt{\lambda}L + h\sin\sqrt{\lambda}L = 0$

$\chi = \sqrt{\lambda}L \qquad \tan\chi = \frac{\chi}{|hL|}$

$\tan\sqrt{\lambda}L = -\frac{\sqrt{\lambda}L}{hL}$

$\Rightarrow \infty$ # e-v $\lambda_n > 0$

$\lambda_n = \frac{\chi_n}{L}$



Can: $\lambda = 0 \qquad \phi = \cancel{c_1} + c_2 x$

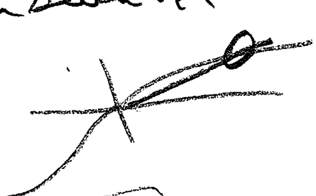
$\phi(0) = 0 = c_1 \qquad \phi_x + h\phi = c_2 + hLc_2 = 0$

$= c_2(1 + hL) = 0 \Rightarrow c_2 = 0 \quad hL = -1$

$\lambda < 0 \qquad \phi = c_1\cosh\sqrt{|\lambda|}x + c_2\sinh\sqrt{|\lambda|}x$

$\phi(0) = 0 = c_1 \qquad \phi(x) = c_2\sinh\sqrt{|\lambda|}x$

$\phi_x + h\phi = c_2\left(\sqrt{|\lambda|}\cosh\sqrt{|\lambda|}L + h\sinh\sqrt{|\lambda|}L\right) = 0$



$\tanh\sqrt{|\lambda|}L = -\frac{\sqrt{|\lambda|}L}{hL} = \frac{\sqrt{|\lambda|}L}{|hL|}$

$\tanh\chi = \chi/|hL|$ are e-v $\boxed{\lambda_1 = -\frac{\chi_1}{L}}$ are ev < 0

$\chi = \sqrt{|\lambda|}L$

4. $W_{tt} = c^2 W_{xx} + F_0 \cos\omega_0 t$

$W_x(0,t) = W_x(L,t) = 0$

$W(x,0) = f(x)$

$W_t(x,0) = 0$

$W = W^{(H)} + W^{(F)}$ $\qquad W_{tt}^{(H)} = c^2 W_{xx}^{H}$ $\qquad W_S^{H} = F(x)G(t)$

$\frac{G_{tt}}{c^2 G} = \frac{F_{xx}}{F} = -\lambda$

$G_{tt} + c^2\lambda G = 0$

$F = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$

$F_x = c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

$F_x(0) = 0 = c_2\sqrt{\lambda}$ $\qquad F_x(L) = 0 = \sqrt{\lambda}\sin\sqrt{\lambda}L = 0$

$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 0, 1, 2, \ldots$

$G = d_1 \cos(c\sqrt{\lambda}t) + d_2 \sin(c\sqrt{\lambda}t)$

$W^{(H)} = \sum_{n=1}^{\infty} \cos\frac{n\pi x}{L}\left(A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right) + A_0$

$W^{(F)} = \Gamma_0 \cos\omega_0 t \Rightarrow$ $\qquad \Gamma_0(-\omega_0^2) = F_0$ $\qquad \omega_0 \neq \omega_* = 0$

$= -\frac{F_0}{\omega_0^2}\cos\omega_0 t$

$w = A_0 + \sum_{1}^{\infty} \cos\frac{n\pi x}{L}\left(A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right) - \frac{F_0}{\omega_0^2}\cos\omega_0 t$

W

$t=0$

$$w(x,0) = f(x) = A_0 - \frac{F_0}{w_0^2} + \sum_1^\infty \cos\frac{n\pi x}{L} A_n$$

$$A_n = \int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \Big/ \underbrace{\int_0^L \cos^2\frac{n\pi x}{L}\,dx}_{=L/2}$$

$$A_0 - \frac{F_0}{w_0^2} = \frac{1}{L}\int_0^L f(x)\,dx$$

$$w_t(x,0) = 0 = \sum_{n=1}^\infty \cos\frac{n\pi x}{L} B_n \frac{n\pi c}{L} \Rightarrow \quad B_n = 0 \quad n=1,2,\dots$$

(b) $w_0 = 0$

$$w_{tt} = c^2 w_{xx} + F_0$$

$$w = w^{(H)} + w^{(F)} \qquad w^{(H)} \text{ same}$$

$$w^{(F)} \Rightarrow \qquad w^F_{tt} = F_0 \qquad w^F = \frac{F_0 t^2}{2}$$

$$w = A_0 + \sum_1^\infty \cos\frac{n\pi x}{L}\left(A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right) + \frac{F_0 t^2}{2}$$

$$w(x,0) = f(x) = A_0 + \sum_1^\infty A_n \cos\frac{n\pi x}{L}$$

$$A_0 = \int_0^L f(x)\,dx / L \qquad A_n = \frac{\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx}{\underbrace{\int_0^L \cos^2\frac{n\pi x}{L}}_{=L/2}}$$

$$w_t(x,0) = 0 = \sum \cos\frac{n\pi x}{L} B_n\left(\frac{n\pi c}{L}\right) \Rightarrow \quad B_n = 0$$

5. $\phi_{xx} + \lambda x^2 \phi = 0 \qquad \phi(1) = \phi(L) = 0$

$\phi = e^{\lambda^p \psi} \qquad \phi_x = \lambda^p \psi_x e^{\lambda^p \psi}$

$\phi_{xx} = (\lambda^{2p} \psi_x^2 + \lambda^p \psi_{xx}) e^{\lambda^p \psi}$

$\lambda^{2p} \psi_x^2 + \lambda^p \psi_{xx} + \lambda x^2 = 0$

$p = 1/2 \qquad \psi_x^2 + \frac{1}{\lambda^{1/2}} \psi_{xx} + x^2 = 0$

$\psi_x = \alpha_0 + \frac{1}{\lambda^{1/2}} \alpha_1 + \cdots \qquad (\alpha_0 + \frac{1}{\lambda^{1/2}} \alpha_1 \cdots)^2 + \frac{1}{\lambda^{1/2}}(\alpha_{0x} + \frac{1}{\lambda^{1/2}} \alpha_{1x}) + x^2 = 0$

$\alpha_0^2 = -x^2 \qquad \alpha_{0\pm} = \pm i x$

$2\alpha_0 \alpha_1 = -\alpha_{0x} \qquad \alpha_1 = -\frac{\alpha_{0x}}{2\alpha_0} = -\frac{1}{2x}$

$\psi_x = \pm i x - \frac{1}{\lambda^{1/2}}\left(\frac{1}{2x}\right) + \cdots$

$\phi_\pm \sim e^{\pm i \frac{x^2}{2}} e^{-\frac{1}{2} \lg x}$

$\phi(x) \sim A \frac{\sin \sqrt{\lambda}\left(\frac{x^2}{2} + \theta_0\right)}{\sqrt{x}} \qquad \phi(1) = 0 \Rightarrow$

$\sin \sqrt{\lambda} \left(\frac{1}{2} + \theta_0\right) = 0$

$\theta_0 = -1/2$

$\phi(x) = \frac{A}{\sqrt{x}} \sin \frac{\sqrt{\lambda}}{2} (x^2 - 1)$

$\phi(L) = 0 = \sin \frac{\sqrt{\lambda}}{2}(L^2 - 1) = 0 \qquad \lambda_n = \left(\frac{2 n \pi}{L^2 - 1}\right)^2$

5. alt

$$\phi_{xx} + \lambda x^2 \phi = 0$$

$$\phi(1) = \phi(L) = 0$$

$$x = 1 + x' \Rightarrow \quad \phi_{x'x'} + \lambda (x'+1)^2 \phi = 0$$

$$\phi(0) = \phi(L-1) = 0$$

$r(x') = (x'+1)^2$: use results from class

$$\phi(x') \sim \frac{1}{[(x'+1)^2]^{1/4}} \sin \sqrt{\lambda} \left(\int_0^{x'} (x'+1)\, dx' + \theta_0 \right)$$

$$\phi(x') \sim \frac{1}{(x'+1)^{1/2}} \sin \sqrt{\lambda} \left(\frac{x'^2}{2} + x' + \theta_0 \right)$$

$$\phi(0) = 0 \Rightarrow \quad \theta_0 = 0$$

$$\phi(x') = \frac{\sin \sqrt{\lambda} \left(\frac{x'^2}{2} + x' \right)}{(x'+1)^{1/2}}$$

$$\phi(L-1) = 0 \quad \sqrt{\lambda_n} \left(\frac{(L-1)^2}{2} + (L-1) \right) = n\pi$$

$$\sqrt{\lambda_n} \left(\frac{L^2}{2} - \frac{1}{2} \right) = n\pi$$

$$\lambda_n = \left(\frac{2n\pi}{L^2 - 1} \right)^2 \quad ✓$$

$$\phi_n(x) = \frac{\sin \sqrt{\lambda_n} \left(\frac{1}{2}(x-1)^2 + (x-1) \right)}{x^{1/2}}$$

$$= \frac{\sin \sqrt{\lambda_n} \frac{(x^2-1)}{2}}{x^{1/2}} \quad ✓$$