APPM 4360/5360 - Review for the final

Material from midterms 1 and 2

1. Multiple valued functions

   - Be able to give examples of multiple valued functions.
   - Know the definition of a branch of a multiple valued function and branch point of a multiple valued function. Be able to give examples.
   - Be able to give examples of properties of roots and logs that hold for real valued functions but do not necessarily hold for their complex versions. For example, is it true in general that for α and β complex,

     $$(z^\alpha)^\beta = z^{\alpha\beta}$$

     or

     $$z^\alpha z^\beta = z^{\alpha+\beta}?$$

     On a specified branch?

   - **Standard question :**

     Be able to compute the roots of a low order polynomial using Euler's formula or some other method.

2. Analyticity

   - **Standard questions :**
     (a) Be able to compute the harmonic conjugate of a real valued harmonic function.
     (b) Be able to determine an analytic function, in terms of $z = x + iy$, from its real and imaginary parts given as functions of x and y.

     $$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

(c) Be able to draw the flow corresponding to an ideal fluid whose complex velocity potential is given by a specified analytic function.

3. Integration

- Know the implications of the Cauchy integral formula
  (a) Liouville's theorem
  (b) Fundamental theorem of algebra
  (c) Gauss's mean value formula
  (d) Maximum/minimum modulus principle
  (e) Maximum/minimum principle for real valued harmonic functions
- **Standard question :**

  Be able to compute a given integral by either parameterization, the fundamental theorem of calculus, the Cauchy-Goursat theorem, the Cauchy integral formula with derivatives, Laurent series, or the residue theorem.

4. Power series

- Know that uniform convergence implies that the limiting function is continuous. Know that uniform convergence allows the function to be integrated and differentiated term by term within its radius of convergence. Know that this does not change the radius of convergence.
- Understand that the Laurent series expansion for a function is unique. That is, if a function has two valid Laurent expansions in a certain region about a specified point, then those two representations are identical. This is what allows us to find Laurent series by manipulating other known series, rather than use the formula for the coefficients.
- **Standard questions :**
  (a) Be able to show that a power series converges uniformly using the Weirstrass M test.

(b) Be able to calculate the radius of convergence of a power series.

(c) By knowing the Taylor series expansion of elementary functions, be able to determine Taylor series for related functions.

(d) Be able to determine the Laurent series expansion for a function about **any** point by suitably manipulating the function (changing variable).

(e) Using Laurent series, be able to classify a singularity as a pole, essential singularity, or removable singularity.

(f) Be able to give the Laurent series expansion of a function in different regions of the complex plane.

(g) Be able to classify singularities at infinity.

New material

1. Residue theory

   - Be able to state and apply the residue theorem.
   - Understand how the residue theorem can be derived from a function's Laurent series expansion.
   - Know how to use the formula for fast calculation of residues at poles

   $$\text{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \Big((z - z_0)^k f(z) \Big)$$

   - Know how to compute the residue at infinity. Understand how the residue at infinity is related to the residues at the poles in the finite complex plane. Know how to apply this idea in the case when a function has many poles in the finite plane and a pole at infinity.

2. Computation of real integrals

   - Using complex integration along the semicircular contour in the upper half plane, know how to compute integrals of the form

   $$\int_{-\infty}^{\infty} f(x)dx$$

where $f(x)$ is either a rational function, $f(x) = g(x)cos(kx)$, or $f(x) = g(x)sin(kx)$.

- Remember the two lemmas proved in class which are essential to these calculations

  1. *If* $|f(z)| \leq Mr^{-k}$ *for* $k > 1$*, then*

  $$\int_{C_r} f(z) \to 0$$

  as $r \to \infty$.

  2. *If* $|f(z)| \leq Mr^{-k}$ *for* $k > 0$*, then*

  $$\int_{C_r} e^{ikz} f(z) \to 0$$

  as $r \to \infty$.

- Know how to compute integrals of the form

  $$\int_0^{2\pi} f(cos(x), sin(x))dx$$

  by using the substitution $z = e^{i\theta}$.

- Given an outline of how to calculate an integral involving a branch point or another miscellaneous integral, be able to justify each step.

3. The argument principle and Rouche's theorem

   - Be able to state the argument principle

     If $f(z)$ *is meromorphic and has* N *zeros and* P *poles inside a simple closed curve* C*, counting multiplicity, then the winding number of* f *around* C*,*

     $$\mathcal{I} = \frac{1}{2\pi i} \int_C \frac{f'}{f} dz,$$

     satisfies $\mathcal{I} = N - P$ *and the change in argument of* f *around* $\mathcal{C}$ *is* $2\pi\mathcal{I}$.

- Be able to state Rouche's theorem

  If $f(z)$ and $g(z)$ are analytic in and on a simple closed curve C with $|f(z)| > |g(z)|$ on C, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.

- Understand how Rouche's theorem can be used to locate the roots of analytic functions.
- Understand how this theorem can be used to locate eigenvalues of matrices.

4. Conformal mappings

   - Be able to define a conformal mapping.
   - Know that an analytic function whose derivative never vanishes defines a conformal mapping.
   - Know what it means for the Laplacian to be invariant under a conformal mappping.
   - Know the statement of the Riemann mapping theorem.
   - Know how to compute the image of a region under a conformal mapping by computing the image of its boundary and the image of an interior point.
   - Be able to give an example of a fractional linear transformation. Know that it maps circles and lines to circles and lines.

5. Applications to Laplace's equation in the plane

   - Understand how Laplace's equation in the plane can be used to model steady state heat flow, electrostatics, or ideal fluid flow.
   - Understand how to solve Dirichlet problems for the above in regions of the plane via the following procedure.

     1. Conformally map the given region to either the upper half plane or unit circle.
     2. Solve the problem in these new domains with the formulas previously derived (these will be given if necessary).
     3. Change variables to the original coordinates.

Matching question

There will be a matching question of the type on the second midterm. You will have to match each given item with one corresponding mathematical statements. The items will be selected from the list below.

- Weirstrass M test
- Fundamental theorem of calculus
- Maximum/minimum modulus principle
- Maximum/minimum principle for real valued harmonic functions
- Cauchy-Goursat theorem
- Morera's theorem
- Cauchy integral formula (derivative form)
- Taylor's theorem
- Laurent's theorem
- The residue theorem
- Liouvlle's theorem
- The argument principle
- Rouche's theorem
- Riemann mapping theorem