2.3 Derivation of Governing Equations

We start by assuming that the incoming wave amplitudes are low enough that their propagation can be approximated as linear - different waves simply superpose on each other. This simplification will quite certainly not hold for 30 meter high giant waves near breaking, but we are here primarily interested not so much in their dynamics once they have built up to giant size, but more in a possible mechanism that can lead to such waves. Let the wave (for now in 1-D) have the form

$$\eta(x,t) = a \cos(k\,x - \sigma\,t) = \text{Re}\; a\; e^{i(k\,x\,-\,\sigma\,t)}\;. \qquad (1)$$

(For simplicity in writing, we will follow the convention of omitting "Re" in most of what follows). All but the first of these following relations are immediate consequences of (1):

$\sigma = \sqrt{g\,k}$	g = acceleration of gravity, approximately 9.8 m/s^2 (relation derived in Section 7.4.1, and assumes deep water)	(2)
$\lambda = 2\,\pi\,/\,k$	λ = wave length ,	(3)
$c_p = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}$	c_p = phase velocity; speed of wave crests increases with λ to reach near-supersonic speeds for earthquake-generated tsunamis with λ up to 600 km - however deep water approximation underlying (4) is then questionable	(4)
$T = 2\,\pi\,/\,\sigma$	T = time period of a wave (as seen from a fixed x-position)	(5)

A typical wave swell might have $T \approx 10$ s, and hence $\lambda \approx 160$ m (swells up to $\lambda \approx 800$ m have been recorded) . The highest possible steady waves can be shown to have $h/\lambda \approx 0.142$ (h being the vertical distance between crest and trough). For $\lambda = 160$ m, the maximal steady height would thus be about 23 meters. That is about 5 times the height that Figure 1 indicates as typical in the ocean south of South Africa. Waves that much lower than the steady ones of maximal height behave reasonably linearly, and (1) is a good enough wave model for studying the onset of focusing.

It is clearly impossible to model the ocean down to individual waves - this would require discretizations into trillions of mesh points, and totally astronomical computer resources (for both time and memory). The first task therefore has to be to identify some 'global' quantity or relationship which can be exploited to provide governing equations, suited for both analysis and numerics.

First, we assume that currents and the wave field (frequencies, amplitudes and directions) do not change in time (at least during the time it takes for waves to travel across the area of interest). The key fact to note is that then, the *time frequency of the wave motion must be the same at all locations*. Surprisingly, this general observation, together with equations (1) and (2) (when generalized to 2-D), contain all the physics that is needed for modeling wave focusing in the presence of arbitrary currents. To see this, we first consider a 1-D case.

2.3.1 Justification of the *constant time frequency* principle.

The time frequency of steady wave oscillations is denoted by $\sigma(k)$. If the water is moving with velocity U, the time frequency seen by a stationary observer would include a Doppler correction, and be $\sigma(k) + k \cdot U$. This is the quantity that we claim will be the same everywhere, i.e. we claim that

$$\sigma(k) + k \cdot U - C \equiv 0 \qquad (6)$$

will hold, where C is some constant. This should be the case even if the medium varies from place to place (i.e. σ is also a function of spatial position x; not just of k), and whatever function $U(x)$ might be. We will make this plausible by looking at a still simpler problem in 1-D:

<u>Example:</u> For the following initial-boundary value problem

PDE: $\frac{\partial u}{\partial t} = -a(x)\frac{\partial u}{\partial x}$ tells that the waves $u(x,t)$ travel with velocity $a(x)$ (>0) to the right,

BC: $u(0,t) = \cos \sigma t$ monochromatic incoming wave swell with time frequency σ; no BC is needed at $x = L$, since the waves move out through this boundary,

IC: None needed we are looking for the wave pattern that establishes itself after any IC has been transported out of the domain,

show that the solution $u(x,t)$ is a purely sinusoidal function of t with frequency σ.

<u>Solution:</u> The algebra becomes a lot simpler if we change the BC to $u(0,t) = e^{i\sigma t}$ - we then just use the real part of the answer we get. By the idea of *separation of variables*, we check if, by any chance, the solution will be of the form

$$u(x,t) = \phi(x)\, e^{i\sigma t} \quad .$$

Substituting into the PDE gives

$$\frac{\partial u}{\partial t} + a(x)\,\frac{\partial u}{\partial x} = i\sigma\,\phi(x)\,e^{i\sigma t} + a(x)\phi'(x)\,e^{i\sigma t} = 0 \qquad ,$$

i.e. $\phi(x)$ satisfies

$$\phi'(x) + \frac{i\sigma}{a(x)}\,\phi(x) = 0 \quad , \quad \phi(0) = 1 \, . \tag{7}$$

This ODE is separable; after dividing through by $\phi(x)$, it can be integrated, giving

$$\ln\,\phi(x) = -i\sigma \int_0^x \frac{d\xi}{a(\xi)} + C_1 \qquad ,$$

i.e.

$$\phi(x) = C_2\, e^{-i\sigma \int_0^x \frac{d\xi}{a(\xi)}}$$

with the condition $\phi(0) = 1$ giving $C_2 = 1$. Thus

$$u(x,t) = C_2\, e^{i\sigma t}\;\; e^{-i\sigma \int_0^x \frac{d\xi}{a(\xi)}} \qquad ,$$

and, returning to the original problem, (taking the real part of this)

$$u(x,t) \;=\; \cos\sigma t\;\; \cos\left\{\sigma \int_0^x \frac{d\xi}{a(\xi)}\right\} \;+\; \sin\sigma t\;\; \sin\left\{\sigma \int_0^x \frac{d\xi}{a(\xi)}\right\} . \tag{8}$$

The time frequency at every point in space is exactly the same, σ, entirely independently of the 'current' $a(x)$.

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Adding a diffusive term $b(x)\,\frac{\partial^2 u}{\partial x^2}$ with $b(x) > 0$ to the PDE will turn (7) to a second order ODE; this will alter $\phi(x)$, but not the time frequency σ. The same will again be the case if we add other types of time independent terms or even if we generalize the 1-D PDE to two space dimensions with arbitrary variable coefficients describing currents - only difference will be that the algebra above will get a lot more complicated.

2.3.2 Application of the constant time frequency principle to 2-D wave field.

In 2-D, $\mathbf{x} = (x_1, x_2)$, and $\mathbf{k} = (k_1, k_2)$ are both vectors. The plane wave

$$\eta(\mathbf{x},t) = e^{\,i\,(\mathbf{k}\cdot\mathbf{x} - \sigma t)} \;=\; e^{\,i\,(k_1\cdot x_1 + k_2\cdot x_2 - \sigma t)}$$

travels in the direction of the (k_1, k_2) - vector. In this direction, it looks like a 1-D wave with

$$k = |\mathbf{k}| = (k_1^2 + k_2^2)^{1/2} \quad .$$

Equation (2) therefore needs to be modified to

$$\sigma = \sqrt{g\,|\mathbf{k}|} \quad . \tag{9}$$

The fundamental equation (6), including a spatially variable current $\mathbf{U}(\mathbf{x})$, becomes

$$H(\mathbf{x},\mathbf{k}) = \sigma(|\mathbf{k}|) + \mathbf{k}\cdot\mathbf{U} - C \equiv 0 \quad . \tag{10}$$

This equation is of precisely the same form as (II4.3.12-13) if we

- let $\mathbf{p} = \mathbf{k}$, and
- note that $\mathbf{k}$ indeed is the gradient of a scalar function

In general, a vector function $\mathbf{k} = [k_1(x_1,x_2),\ k_1(x_1,x_2)]$ will not be the gradient of some single-valued scalar function. But here, we are assured of this since the 2-D wave is $\eta(\mathbf{x},t) = a\, e^{i(\mathbf{k}\cdot\mathbf{x} - \sigma t)}$, i.e. $\mathbf{k}$ represents the gradient of the phase angle of the wave.

2.3.3 Application of the Hamiltonian equations to the constant frequency equation

The constant frequency equation is a special case the Hamiltonian discussed in Section II.4.3. Figure 1 compares the notation in the general and the present case.

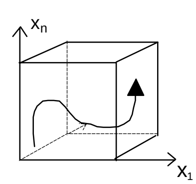
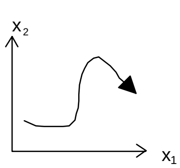
Case		Conservation law	ODE system
n - D General Hamiltonian equation		$F(x_1, x_2, ..., x_n, p_1, p_2, ..., p_n) = 0$ where $p_i = \frac{\partial u}{\partial x_i},\ i = 1, 2, ..., n.$	$\begin{cases} \frac{\partial x_i}{\partial s} = \frac{\partial F}{\partial p_i} \\ \frac{\partial p_i}{\partial s} = -\frac{\partial F}{\partial x_i} \end{cases}$ $i = 1, 2, ..., n$
2-D Wavefield in water with currents		$H(x_1, x_2, k_1, k_2) = 0$ where $H \equiv \sigma(\lvert\mathbf{k}\rvert) + \mathbf{k}\cdot U$; $k_i = \frac{\partial\phi}{\partial x_i}, i = 1, 2.$; with ϕ the wave phase speed.	$\begin{cases} \frac{\partial x_i}{\partial t} = \frac{\partial H}{\partial k_i} \\ \frac{\partial k_i}{\partial t} = -\frac{\partial H}{\partial x_i} \end{cases}$ $i = 1, 2.$

Figure 1. Comparison of general Hamiltonian equation and special case arising in water wave context.

As we see in Figure 1, to create the ODEs corresponding to (10), we need to form the quantities

$$\left\{\begin{array}{l} \frac{\partial H}{\partial k_1} = \alpha k_1 + U_1 \\ \frac{\partial H}{\partial k_2} = \alpha k_2 + U_2 \\ \frac{\partial H}{\partial x_1} = k_1 \frac{\partial U_1}{\partial x_1} + k_2 \frac{\partial U_2}{\partial x_1} \\ \frac{\partial H}{\partial x_2} = k_1 \frac{\partial U_1}{\partial x_2} + k_2 \frac{\partial U_2}{\partial x_2} \end{array}\right. \quad \text{with } \alpha = \frac{d\sigma}{d\,|\mathbf{k}|} = \frac{\sqrt{g}}{2\,|\mathbf{k}|^{3/2}} \quad \text{according to (2)}$$

Hence, the system of equations we need to solve in order to find **k**(**x**) - the complete wave state - is

$$\left\{\begin{array}{llll} & \text{With no current} & \text{Terms due to current} & \\ \frac{\partial x_1}{\partial t} = & \alpha k_1 & + U_1 & \\ \frac{\partial x_2}{\partial t} = & \alpha k_2 & + U_2 & \\ \frac{\partial k_1}{\partial t} = & & -\left(k_1 \frac{\partial U_1}{\partial x_1} + k_2 \frac{\partial U_2}{\partial x_1}\right) & \\ \frac{\partial k_2}{\partial t} = & & -\left(k_1 \frac{\partial U_1}{\partial x_2} + k_2 \frac{\partial U_2}{\partial x_2}\right) & \end{array}\right. . \qquad (11)$$

The independent variable t is for the moment just a parameter (will turn out to correspond to physical time). What the system (11) tells is that, if **U**(**x**) is given, and we at some location **x** = (x_1,x_2) specify an initial wave (by giving **k** = (k_1,k_2) there), solving the four coupled ODEs (11) forward in t will trace out a path in the (x_1,x_2)-plane along which we simultaneously obtain k_1 and k_2 . Starting from many places along a domain boundary, we can trace out a dense network of paths - and hence obtain k_1 and k_2 throughout a domain. Each such path represents how a *ray* progresses - a natural concept in case of light, but for water a more abstract concept; a path that is orthogonal to the wave fronts. Wave *energy* follows these rays - if rays cross each other, wave energy has focused.

We can note that in the case of no current, k_1 and k_2 remain constant, and rays are straight.

Well hidden in the interpretation of the rays are a few physical assumptions than have not yet been pointed out. A notable one is that the wavelength λ must be small in comparison with any objects or flow features (such as currents). This tends to be the case for light, making rays a particularly clear concept. It is much less clear in the case of sound - such waves travels easily around everyday objects. Another key difference between the linear propagation of light vs. the non-linear of water waves is that the former can cross each other without any interference. Large water waves coming together are likely to break, and therefore change their character. The ray model is mainly useful to highlight areas with potentially high wave energies, but it can not be expected to give any details within or after waves have passed through focusing areas.