4.5 Error Analysis

There are many sources of errors encountered in the GPS process. A recording of a position is not of much use if it is not accompanied by some form of error estimate. The following is a very schematic summary of how much different sources typically contribute:

Typical errors (in meters) in computed distance to each satellite due to different error sources

Source	Standard GPS	Differential GPS
Satellite Clocks	1.5	0
Orbit Errors	2.5	0
Ionospheric delays	5.0	0.4
Tropospheric delays	0.5	0.2
Receiver noise	0.3	0.3
Multipath	0.6	0.6
Selective availability (SA)	30	0
Typical resulting positional accuracy		
Horizontal	50	1.3
Vertical	78	2.0
3-D	93	2.8

We will here carry out one example of error analysis to illustrate the process of tracing how different sources of errors in input data can carry through to errors in the computed position. One key feature this will illustrate is that error analysis generally is *linear*; the final effect of different error sources can be studied separately, and effects can be added together for a 'worst case' estimate.

If there are many measurements available for the same quantity, some errors will typically fluctuate between measurements and partly cancel if one considers an average of the readings. In this scenario, 'worst case' estimates are unduly pessimistic, and statistical tools should be applied. For example, with n estimates, the expected error often decreases like $1/\sqrt{n}$.

We suppose here that we have only the top four equations of the set I.4.4-(1) available. We write these in the form

$$f_1 \equiv \quad (x - 3)^2 + (y - 2)^2 + (z - 3)^2 - [\,(T_1 + t_1 - t) \cdot c]^2 = 0 \qquad (1\text{ a})$$
$$f_2 \equiv \quad (x - 1)^2 + (y - 3)^2 + (z - 1)^2 - [\,(T_2 + t_2 - t) \cdot c]^2 = 0 \qquad (1\text{ b})$$
$$f_3 \equiv \quad (x - 5)^2 + (y - 7)^2 + (z - 4)^2 - [\,(T_3 + t_3 - t) \cdot c]^2 = 0 \qquad (1\text{ c})$$
$$f_4 \equiv \quad (x - 1)^2 + (y - 7)^2 + (z - 3)^2 - [\,(T_4 + t_4 - t) \cdot c]^2 = 0 \qquad (1\text{ d})$$

where the recorded time delays according to the receiver's very inaccurate clock are

T_1 = 10010.00692286
T_2 = 10013.34256381
T_3 = 10016.67820476
T_4 = 10020.01384571 .

We have here also introduced additional variables t_1, t_2, t_3, t_4 which represent further errors in the timing signals from each of the four satellites (causes for these could for example be ionospheric delays). We want to estimate what the uncertaincy is in position x, y, z and corrected time t are as functions of variations in t_1, t_2, t_3, and t_4.

Simplified one variable / one equation situation:

Had our system of equations been just one scalar equation in one variable

$$f_1 \equiv \quad (x - 3)^2 - [\,(T_1 + t_1) \cdot c]^2 = 0 \qquad (2)$$

we would first set $t_1 = 0$ (i.e. assuming this extra error was not there) and solve for x. Next, we would re-introduce t_1 and ask how much variations in that will influence x. Hence, we view x as a function of t_1: $x = x(t_1)$. Differentiating (2) with repect to t_1 gives

$$\frac{df_1}{dt_1} = 2(x-3)\,\frac{dx}{dt_1} - 2c^2(T_1 + t_1) = 0$$

Now we again set $t_1 = 0$ and solve for $\frac{dx}{dt_1}$. That derivative is precisely what we want - a measure of how much x will change for small changes in t_1.

Original four variable / four equation situation:

In (1 a-d), we similarly have $x = x(t_1,t_2,t_3,t_4)$, $y = y(t_1,t_2,t_3,t_4)$, $z = z(t_1,t_2,t_3,t_4)$, and $t = t(t_1,t_2,t_3,t_4)$. Differentiating f_i with respect to t_j becomes an exercise in using the chain rule:

$$\frac{df_i}{dt_j} = \frac{\partial f_i}{\partial x}\frac{\partial x}{\partial t_j} + \frac{\partial f_i}{\partial y}\frac{\partial y}{\partial t_j} + \frac{\partial f_i}{\partial z}\frac{\partial z}{\partial t_j} + \frac{\partial f_i}{\partial t}\frac{\partial t}{\partial t_j} + \frac{\partial f_i}{\partial t_j} = 0 \ , \qquad i, j = 1,\ldots,4$$

This is most clearly written in martix form

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial t} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial t} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} & \frac{\partial f_3}{\partial t} \\ \frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial z} & \frac{\partial f_4}{\partial t} \end{bmatrix} \times \begin{bmatrix} \frac{\partial x}{\partial t_1} & \frac{\partial x}{\partial t_2} & \frac{\partial x}{\partial t_3} & \frac{\partial x}{\partial t_4} \\ \frac{\partial y}{\partial t_1} & \frac{\partial y}{\partial t_2} & \frac{\partial y}{\partial t_3} & \frac{\partial y}{\partial t_4} \\ \frac{\partial z}{\partial t_1} & \frac{\partial z}{\partial t_2} & \frac{\partial z}{\partial t_3} & \frac{\partial z}{\partial t_4} \\ \frac{\partial t}{\partial t_1} & \frac{\partial t}{\partial t_2} & \frac{\partial t}{\partial t_3} & \frac{\partial t}{\partial t_4} \end{bmatrix} = -\begin{bmatrix} \frac{\partial f_1}{\partial t_1} & \frac{\partial f_1}{\partial t_2} & \frac{\partial f_1}{\partial t_3} & \frac{\partial f_1}{\partial t_4} \\ \frac{\partial f_2}{\partial t_1} & \frac{\partial f_2}{\partial t_2} & \frac{\partial f_2}{\partial t_3} & \frac{\partial f_2}{\partial t_4} \\ \frac{\partial f_3}{\partial t_1} & \frac{\partial f_3}{\partial t_4} & \frac{\partial f3}{\partial t_3} & \frac{\partial f_3}{\partial t_4} \\ \frac{\partial f_4}{\partial t_1} & \frac{\partial f_1}{\partial t_2} & \frac{\partial f_4}{\partial t_3} & \frac{\partial f_4}{\partial t_4} \end{bmatrix}$$

Taking partial derivatives of (1 a-d) gives all the entries of the first and last matrices above

$$2\begin{bmatrix} x-3 & y-2 & z-3 & c^2(T_1+t_1-t) \\ x-1 & y-3 & z-1 & c^2(T_2+t_2-t) \\ x-5 & y-7 & z-4 & c^2(T_3+t_3-t) \\ x-1 & y-7 & z-3 & c^2(T_4+t_4-t) \end{bmatrix} \times \begin{bmatrix} \frac{\partial x}{\partial t_1} & \frac{\partial x}{\partial t_2} & \frac{\partial x}{\partial t_3} & \frac{\partial x}{\partial t_4} \\ \frac{\partial y}{\partial t_1} & \frac{\partial y}{\partial t_2} & \frac{\partial y}{\partial t_3} & \frac{\partial y}{\partial t_4} \\ \frac{\partial z}{\partial t_1} & \frac{\partial z}{\partial t_2} & \frac{\partial z}{\partial t_3} & \frac{\partial z}{\partial t_4} \\ \frac{\partial t}{\partial t_1} & \frac{\partial t}{\partial t_2} & \frac{\partial t}{\partial t_3} & \frac{\partial t}{\partial t_4} \end{bmatrix} =$$

$$= 2\begin{bmatrix} c^2(T_1+t_1-t) & 0 & 0 & 0 \\ 0 & c^2(T_2+t_2-t) & 0 & 0 \\ 0 & 0 & c^2(T_3+t_3-t) & 0 \\ 0 & 0 & 0 & c^2(T_4+t_4-t) \end{bmatrix}$$

Writing this as $A\,X = B$, we can solve for X by simply multiplying by A^{-1} from the left (or better still - view this as a linear system of equations with four RHSs and four side-by-side solution vectors. Using the known values for T_i , setting $t_i = 0$ and substituting in our numerical solution $x = 5$, $y = 3$, $z = 1$ and $t = 10{,}000$ gives

$$\begin{bmatrix} \frac{\partial x}{\partial t_1} & \frac{\partial x}{\partial t_2} & \frac{\partial x}{\partial t_3} & \frac{\partial x}{\partial t_4} \\ \frac{\partial y}{\partial t_1} & \frac{\partial y}{\partial t_2} & \frac{\partial y}{\partial t_3} & \frac{\partial y}{\partial t_4} \\ \frac{\partial z}{\partial t_1} & \frac{\partial z}{\partial t_2} & \frac{\partial z}{\partial t_3} & \frac{\partial z}{\partial t_4} \\ \frac{\partial t}{\partial t_1} & \frac{\partial t}{\partial t_2} & \frac{\partial t}{\partial t_3} & \frac{\partial t}{\partial t_4} \end{bmatrix} = \begin{bmatrix} 0.1499 & -0.3498 & -0.6246 & 0.8244 \\ 0.1499 & 0.2498 & 0.1249 & -0.5246 \\ -0.4497 & 0.7495 & 0.3747 & -0.6745 \\ -0.5000 & 2.1667 & 2.0833 & -2.7500 \end{bmatrix} \qquad (3)$$

This tells how sensitive each variable x, y, x, t is to the small errors in the timings t_1, t_2, t_3, t_4 for the signals from the four satellites.

If the timings are all accurate to within 0.1 μs = 0.0001 ms, the worst case errors in the result would be

x-dir: (0.1499 + 0.3498 + 0.6246 + 0.8244) · 0.1 km ≈ 195 meters
y-dir: (0.1499 + 0.2498 + 0.1249 + 0.5246) · 0.1 km ≈ 105 meters
z-dir: (0.4497 + 0.7495 + 0.3747 + 0.6745) · 0.1 km ≈ 225 meters
t-err: (0.5000 + 2.1667 + 2.0833 + 2.7500) · 0.1 μs ≈ 0.75 μs

In this case, the positional error turns out to be least in the *y*- direction. The best time the receiver can calculate is about 7.5 times less accurate than the precision of the incoming time signals.

This analysis above was based on reception from only the first four of our six satellites, giving us a 4×4 matrix in (3) with sensitivity information. In a similar way, we could have analyzed the full 6-satellite case to arrive at

$$\begin{bmatrix} \frac{\partial x}{\partial t_1} & \frac{\partial x}{\partial t_2} & \frac{\partial x}{\partial t_3} & \frac{\partial x}{\partial t_4} & \frac{\partial x}{\partial t_5} & \frac{\partial x}{\partial t_6} \\ \frac{\partial y}{\partial t_1} & \frac{\partial y}{\partial t_2} & \frac{\partial y}{\partial t_3} & \frac{\partial y}{\partial t_4} & \frac{\partial y}{\partial t_5} & \frac{\partial y}{\partial t_6} \\ \frac{\partial z}{\partial t_1} & \frac{\partial z}{\partial t_2} & \frac{\partial z}{\partial t_3} & \frac{\partial z}{\partial t_4} & \frac{\partial z}{\partial t_5} & \frac{\partial z}{\partial t_6} \\ \frac{\partial t}{\partial t_1} & \frac{\partial t}{\partial t_2} & \frac{\partial t}{\partial t_3} & \frac{\partial t}{\partial t_4} & \frac{\partial t}{\partial t_5} & \frac{\partial t}{\partial t_6} \end{bmatrix} = \begin{bmatrix} -0.0362 & -0.1061 & -0.0267 & 0.2221 & -0.4185 & 0.3655 \\ 0.1240 & 0.2097 & -0.1619 & -0.3125 & 0.2007 & -0.0602 \\ -0.0643 & 0.3353 & 0.0169 & -0.0456 & 0.2505 & -0.6213 \\ -0.3616 & 1.1765 & 0.0047 & -0.5127 & 1.4550 & -1.4852 \end{bmatrix}$$

This time the worst-case errors are notably smaller even though none of the inputs are any more accurate; they are

x-dir: ≈ 118 meters
y-dir: ≈ 107 meters
z-dir: ≈ 133 meters
t-err: ≈ 0.50 μs

The improvement in expected errors is better still - the probability that the sign and size of all errors to conspire to create a maximum error situation is far less likely the more independent input variables that enter. Cancellation of errors becomes increasingly likely.