1.2 Dimensional analysis

There are two main steps in 'non-dimensionalizing' an equation:

<u>Step 1:</u> List its variables and parameters, and note what type of units they are expressed in (such as $\mathcal{L}$ for length or $\mathcal{T}$ for time):

Variables		Dimension
Dependent variable y^*		$\mathcal{L}$
Independent variable t^*		$\mathcal{T}$
Parameters		**Dimension**
Gravitational acceleration g		$\mathcal{L}\mathcal{T}^{-2}$
Initial speed V		$\mathcal{L}\mathcal{T}^{-1}$
Radius of earth R		$\mathcal{L}$
Mass M	(quantity disappeared while deriving (1))	$\mathcal{M}$

<u>Step 2:</u> For each of the two variables, find some combination of the parameters that produce the same dimensions. For example

y^* has dimension $\mathcal{L}$; this is the same as that of R, and
t^* has dimension $\mathcal{T}$; the combination R/V becomes also $\mathcal{T}$.

These choices were not unique. Although they will allow us to non-dimensionalize the equation, we will soon see that we did not get the equation in its most practical form - that is the topic of the next section discussing *scaling*.

Based on these choices, we introduce new (dependent and independent) variables

$$y = \frac{y^*}{R} \qquad \text{and} \qquad t = \frac{t^*}{{}^{-1}} \quad . \tag{1}$$

These variables are now non-dimensional, i.e. their numerical value do not depend on what kind of units were originally used (e.g. meters, inches, nautical miles or light years for distances and seconds, weeks or millennia for time).

<u>Step 3:</u> Express the original equation (II1.1-1) in the new variables (from Step 2). To do this, we note first how a derivative changes if its dependent and independent variables are *scaled*, such as through (1):

If $y^* \to \alpha y$, $t^* \to \beta t$, then $\frac{dy^*}{d(t^*)} \to \frac{\alpha}{\beta} \cdot \frac{dy}{dt}$, $\frac{d^2 y^*}{d(t^*)^2} \to \frac{\alpha}{\beta^2} \cdot \frac{d^2 y}{dt^2}$, etc.; in general $\frac{d^p y^*}{d(t^*)^p} \to \frac{\alpha}{\beta^p} \cdot \frac{d^p y}{dt^p}$. The placement of the p's in the (curious) notation $\frac{d^p y}{dt^p}$ for a p'th derivative makes this easy to remember - the scale change for the variable in the numerator just factors right out, whereas the one for the variable in the denominator comes out raised to the p'th power.

Hence (after slight algebra), our re-scaling changes (II1.1-1) into

$$\varepsilon \frac{d^2 y}{dt^2} = -\frac{1}{(y+1)^2} \;, \qquad y(0) = 0,\; \frac{dy}{dt}(0) = 1 \tag{2}$$

where

$$\varepsilon = \frac{V^2}{gR} \quad . \tag{3}$$

Through these three steps, we have now achieved:

1. All variables in (2) are non-dimensional - there are no remains left from the choice of measuring units that was originally used (in particular, note that the 'dimension' of ε works out as $(\mathcal{L}\mathcal{T}^{-1})^2/(\mathcal{L}\mathcal{T}^{-2}\mathcal{L}) = \mathcal{L}^0\mathcal{T}^0$, i.e. ε is also non-dimensional).

2. The form of (2) is clearly simpler than (8.1-1) for both analysis and numerics, and

3. Most importantly, we have discovered that the problem can be formulated using only one independent parameter (ε), not three (V, g and R). This makes it much easier to explore the complete parameter space of solutions. Once (2) and (3) have been approximated by some means (in rare cases analytically, else with perturbation- or numerical methods), we can use (1) to return to the original physical variables.

A very important consequence of this is that it makes 'scale models' possible. These tells how differently scaled 'set-ups' can be mathematically equivalent, allowing full-scale scenarios to be tested accurately in much smaller sizes, e.g. ship models in tow tanks or airplane models in wind tunnels (Typically, a problem will contain more than one non-dimensional parameter, but the opportunities of making simplifying scale changes persist).

To arrive at a single combined parameter such as ε defined in (3), it is not even necessary to know the equation that governs the process we are interested in. A general combination $g^\alpha \cdot V^\beta \cdot R^\gamma$ will get dimension $\mathcal{L}^\alpha\mathcal{T}^{-2\alpha}\,\mathcal{L}^\beta\mathcal{T}^{-\beta}\,\mathcal{L}^\gamma = \mathcal{L}^{\alpha+\beta+\gamma}\mathcal{T}^{-2\alpha-\beta}$. This can only be non-dimensional if $\begin{cases} \alpha+\beta+\gamma &= 0 \\ -2\alpha-\beta &= 0 \end{cases}$, an underdetermined linear system in which we can choose γ arbitrary, and then find $\alpha = \gamma$, $\beta = -2\gamma$. The only possible non-dimensional combinations of g, V and R are therefore of the form $\left(\frac{gR}{V^2}\right)^\gamma$ with γ arbitrary. If our primary interest lies in the limit of V small, it becomes most natural to choose $\gamma = -1$, and thus arriving again at (3). Even if we do not make that choice ($\gamma = -1$), we still see that the solution will depend on just one particular combination of g, r, V.

As we noted, step 2 was not unique. Instead of

$$" \quad t^* \quad \text{has dimension } \mathcal{T}; \quad \text{the combination } R/V \text{ becomes also } \mathcal{T} \quad ",$$

we could have noted

$$t^* \quad \text{has dimension } \mathcal{T}; \quad \text{the combination } \sqrt{R/g} \text{ becomes also } \mathcal{T}.$$

Instead of arriving at (2) , this alternative gives (again, after slight algebra)

$$\frac{d^2y}{dt^2} = -\frac{1}{(y+1)^2} \quad , \quad y(0) = 0, \; \frac{dy}{dt}(0) = \varepsilon^{1/2} \;, \qquad (4)$$

with ε unchanged from (3). There are still other possibilities for non-dimensionalizing in Step 2.

Regarding the three main techniques to handle the ODE, the significance of non-dimensionalization can be summarized as follows:

- Analytic solution — Scaled problem 'cleaner', but not genuinely any easier,
- Numeric solution — Same situation as above, and
- Asymptotic approx. — Often used idea is to expand in a series as $\varepsilon \to 0$. Here, non-dimensionalization makes a big difference, and *scaling* produces the most favorable choice for Step 2 above.

  As it will transpire in the next section, neither (2), nor (4) represent the best forms of the ODE. Scaling will lead us to a third form.