

## 1.3 Scaling

Scaling addresses the ambiguity we found in how to non-dimensionalize the dependent and independent variables ( $y^*$  and  $t^*$ ). Although both of the versions we considered were successful in simplifying the ODE and in identifying how the independent parameters could be consolidated into one single one ( $\epsilon = \frac{V^2}{gR}$ ), they are unsatisfactory if we want to apply the very powerful techniques known as perturbation methods. These allow the present, difficult-to-solve ODE, to be replaced by a series of simpler ones, whose solutions provide successive terms  $y_k(t)$  in expansions of the form

$$y(t, \epsilon) = \sum_{k=0}^N \epsilon^k y_k(t)$$

A useful expansion of this type requires, as a minimum, that the solution for  $\epsilon = 0$  makes sense as a limit of  $\epsilon \rightarrow 0$ . Putting  $\epsilon = 0$  gives in the first case (III.2-2) the impossible requirement

$$0 = -\frac{1}{(y+1)^2}, \quad y(0) = 0,$$

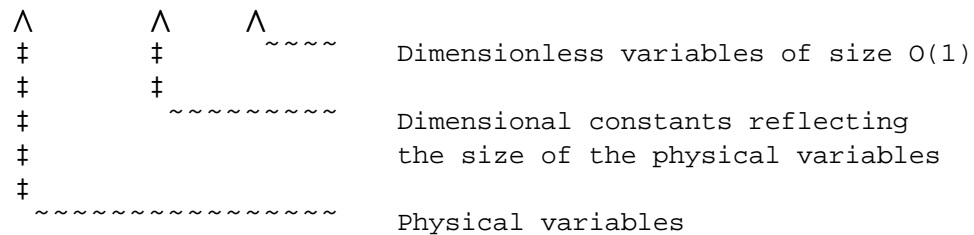
and in the second case (III.2-3)

$$\frac{d^2y}{dt^2} = -\frac{1}{(y+1)^2}, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = 0$$

implying that  $y(t) < 0$  for all  $t > 0$ , i.e. the projectile will never rise at all. Both cases are obviously nonsensical. The problem is that, as  $\epsilon \rightarrow 0$ , the non-dimensionalization also causes  $y$  and  $t$  to change in such ways that the time- and space ranges we are interested in get stretched out or compressed away, rather than converging to some fixed ranges.

We need the non-dimensionalization to have the following structure:

$$\begin{aligned} y^* &= c_y \cdot y \\ t^* &= c_t \cdot t \end{aligned}$$



This requires some physical arguments:

1. *Decide what parameter regime we are interested in:*

At present, this is  $V$  small, i.e.  $\epsilon$  small,

2. *In the simplified case of  $\epsilon$  small, find dimensional constants that are proportional to the physical variables:*

With  $V$  small, the projectile will not reach very high, and the acceleration of gravity becomes approximately  $g$  (constant). We can then easily solve for the height that will be reached ( $\frac{1}{2} V^2/g$ ) and the time this will take ( $V/g$  for each of the up- and down paths). Not needing to be concerned about numerical constants, this suggests choosing  $c_y = V^2/g$  and  $c_t = V/g$ .

3. *Carry out the change of variables:*

$$\text{We get } \frac{d^2y}{dt^2} = -\frac{1}{(1 + \epsilon y)^2}, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = 1 \quad (1)$$

(as before with  $\epsilon = \frac{V^2}{gR}$ ).

Finally, we have a formulation in which letting  $\epsilon \rightarrow 0$  causes  $y(t, \epsilon)$  to converge, and  $y^*(t^*)$  to recover the physical behavior of this limit. The equation (1) is ideally suited for perturbation analysis.

It is quite common that equations are encountered in an already non-dimensionalized form. A first task is then to check whether some re-scaling is needed, or if  $\epsilon = 0$  indeed corresponds to  $\epsilon \rightarrow 0$ . It is also common that no single re-scaling is appropriate for the complete spatial domain. One such case of 'singular' perturbation concerns boundary layers in fluids. Different scalings need to be used at a boundary and throughout the bulk of the domain. Separately scaled perturbation expansions from these two (or more) regions have to be 'matched' to agree where they spatially meet. In the next chapter, we focus only on 'regular' perturbation problems, in which one single scaling suffices for the entire problem.