Chapter 4

FOURIER SERIES / TRANSFORMS

We will discuss four main versions of Fourier series / transforms - all closely related with each other. These four are sketched out in Figure 1:

	Type	Domain	Description
i.	Continuous,	$[-\infty,\infty]$	Fourier transform
ii.	Continuous,	$[-\pi, \pi]$	Fourier series
iii.	Discrete,	$[-\pi, \pi]$	DFT - discrete Fourier transform
iv.	Discrete,	$[0, N]$	DFT - case handled effectively by the FFT algorithm.

Of these, case i. arises in the theory for CT (computerized tomography - Chapter 2). Effective numerical implementation of the Fourier technique for CT relies on case iv, and the FFT algorithm (described in Chapter III.5). The order of the four cases in Figure 1, i-iv, reflect that, in some sense, i can be seen as the most fundamental case, of which ii, iii, iv are successively more restricted special cases. The conceptually simplest case is probably case ii - the Fourier series of a periodic function. Hence, that case is described first, in Section 4.1. The subsequent Sections 4.2 and 4.3 cover the remaining cases i and iii with iv respectively. Generalizations to 2-D are discussed in Section 4.4

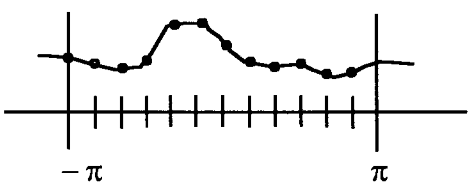
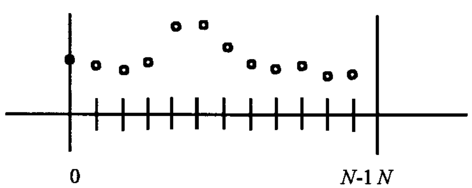
TYPE OF DATA In all cases, assumed complex	FORM OF FOURIER REPRESENTATION	TRANSFORM Formulas for Fourier coefficients
CONTINUOUS, $(-\infty, \infty)$  	$u(x) = \int_{-\infty}^{\infty} \hat{u}(\omega)\, e^{i\omega x}\, d\omega$ FOURIER TRANSFORM EVERY mode $e^{i\omega x}, -\infty < \omega < \infty$ possible	$\hat{u}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x)\, e^{-i\omega x}\, dx$
CONTINUOUS, $(-\pi, \pi)$  	$u(x) = \sum_{k=-\infty}^{\infty} \hat{u}_k e^{ikx}$ FOURIER SERIES ONLY modes that are 2π–periodic are possible	$\hat{u}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x)\, e^{-ikx}\, dx$
DISCRETE, $(-\pi, \pi)$, N points  	$u(x) = \sum_{k=0}^{N-1} \hat{u}_k\, e^{ikx}$ DISCRETE FORUIER TRANSFORM ONLY N modes present	$\hat{u}_k = \frac{1}{N} \sum_{j=0}^{N-1} u(x_j)\, e^{-ikx_j}$ (Grid points $x_j = -\pi + j \cdot 2\pi/N$, $j = 0, 1, \ldots, N-1$)
DISCRETE, $(0, N)$, N points  	$u_j = \sum_{k=0}^{N-1} \hat{u}_k e^{2\pi ikj/N}$ DISCRETE FOFOURIER TRANSFORM N modes (case handled by FFT algorithm)	$\hat{u}_k = \frac{1}{N} \sum_{j=0}^{N-1} u_j e^{-2\pi ikj/N}$ (case handled by FFT algorithm)

Figure 1. Comparison between different types of Fourier expansions.