3.1 Fourier series

With very mild conditions on a [-π,π]-periodic function $u(x)$, it can be written as a sum of all the sines and cosines that also are periodic over the same interval. This result, due to Joseph Fourier (1768-1830) is one of the most fundamental ones in mathematics:

$$u(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx \quad . \qquad (1)$$

Supposing that this is true, it is then very easy to find the appropriate coefficients a_0 and a_k, b_k , k = 1,2,... For example, to find b_n for some integer n, we consider

$$\int_{-\pi}^{\pi} u(x) \sin nx\, dx = a_0 \int_{-\pi}^{\pi} \sin nx\, dx + \sum_{k=1}^{\infty} a_k \int_{-\pi}^{\pi} \cos kx \sin nx\, dx + \sum_{k=1}^{\infty} b_k \int_{-\pi}^{\pi} \sin kx \sin nx\, dx \qquad (2)$$

Because of

$$\cos kx \sin nx = \tfrac{1}{2} \sin (k+n)x - \tfrac{1}{2} \sin (k-n)x \quad , \qquad \text{and}$$
$$\sin kx \sin nx = -\tfrac{1}{2} \cos (k+n)x + \tfrac{1}{2} \cos (k-n)x \quad ,$$

every term in the right hand side of (2) integrates to zero apart from the one term which corresponds to $\frac{1}{2} \cos (k-n)x \equiv \frac{1}{2}$ in the case of $k = n$. Hence $\int_{-\pi}^{\pi} u(x) \sin nx\, dx = b_k \cdot \pi$, giving the value of b_k. The coefficients a_0 and a_k are found similarly:

$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x)\, dx \\ a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} u(x) \cos kx\, dx \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} u(x) \sin kx\, dx \end{aligned}\right\} \;, \; k = 1, 2, \ldots \quad . \qquad (3)$$

It is often inconvenient to have three types of terms in (1), and correspondingly, the three different formulas above for the coefficients. By Euler's equation

$$e^{ix} = \cos x + i \sin x$$

which implies

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad , \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad ,$$

we can also express $u(x)$ as

$$u(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \quad . \qquad (4)$$

Instead of the three equations (3), we get

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) e^{-ikx} dx \quad , k = -\infty, \ \ldots, \infty. \qquad (5)$$

Equations (4) and (5) correspond to case 2 "Continuous, (-π,π)" shown in Figure III4.0-1.

In the usual situation of $u(x)$ real, c_0 is real, and $c_{-k} = \overline{c_k}$, k = 1, 2, ... , ∞ (with the bar denoting complex conjugate):

$$c_{-k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) e^{ikx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) \overline{e^{-ikx}} dx = \overline{\frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) e^{-ikx} dx} = \overline{c_k} \qquad (6)$$

.

If $u(x)$ has a jump-discontinuity, its Fourier series will, at the jump, converge to the average value for the two sides. Near the jump, a *Gibbs' phenomenon* occurs: Partial sums $\Sigma_{k=-N}^{N} c_k e^{ikx}$ will at each side 'overshoot' the jump by about 9 %. As $N \to \infty$, the region in which this occurs gets narrower, but the height persists. In this case, the error of the partial sum - in max-norm - will be O(1) near the jump, and O(1/N) away from it. If $u(x)$ is smoother, convergence rates become faster:

Order of max-norm errors in partial Fourier sums caused by irregularities of a function

	Max-norm of errors (order)	
Function	Near irregularity	Away from irregularity
f discont.	1	$1/N$
f' discont.	$1/N$	$1/N^2$
f'' discont.	$1/N^2$	$1/N^3$
⋮	⋮	⋮
f analytic (periodic)	e^{-cN}, $c > 0$	

Another version of the Gibbs' phenomenon occurs if one performs equi-spaced interpolation with trigonometric functions. In this case, the overshoot approaches 14 % of the height of the jump as m - denoting the number of interpolation points - tends to infinity. Figure 1 illustrates this latter case (but marks also the limiting overshoot height in the case of a truncated infinite expansion).

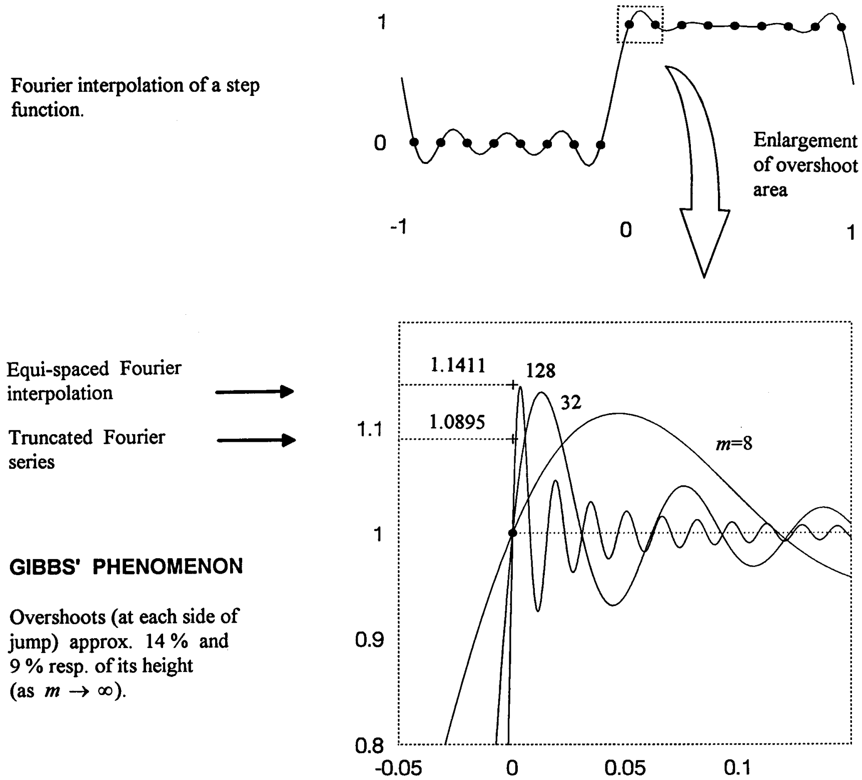

Figure 1. Gibbs' phenomenon for trigonometric interpolation and for truncated Fourier series.

The result that a Fourier series converges if $u(x)$ is piecewise differentiable can be weakened somewhat, but attempting to do this leads to surprisingly tricky mathematics.

In spite of the fact that a truncated Fourier series $\sum_{k=-N}^{N} c_k e^{ikx}$ always gives the best approximation to $u(x)$ of all possible sums $\sum_{k=-N}^{N} d_k e^{ikx}$ in the sense of least squares, i.e. $\int_{-\pi}^{\pi}(u(x)-\sum_{k=-N}^{N} d_k e^{ikx})^2\,dx$ is minimized, the Fourier series of a continuous function $u(x)$ can still diverge to infinity at one (or more) points as $N \to \infty$. However, such mathematical subtleties play no role in the utility of Fourier series in modeling.

It is always permissible to integrate (term-by-term) both sides of the Fourier expansions (1) or (4); taking derivatives works only on condition that the resulting series converge.

The uses of Fourier expansions in analysis / modeling / engineering are too many to even start to list. In Section 4.2, it is used to solve a simple 1-D initial-boundary value problem for the heat equation.