3.2 Fourier transform

To keep the notation as simple as possible, we introduced Fourier series on the interval [-π,π]. We could as well have considered a general [-*L*π,*L*π]. With the additional change of calling the function $u(x)$ and the coefficients $\hat{u}_k$, equations (II3.1-4) and (II3.1-5) would become replaced by

$$u(x) = \sum_{k=-\infty}^{\infty} \hat{u}_k\, e^{i\frac{k}{L}x} \qquad \text{where} \qquad \hat{u}_k = \frac{1}{2\pi L}\int_{-L\pi}^{L\pi} u(x)\,e^{-i\frac{k}{L}x}\,dx \quad .$$

Writing $\hat{u}_k = \frac{1}{L}\hat{u}\left(\frac{k}{L}\right)$ gives

$$u(x) = \frac{1}{L}\sum_{k=-\infty}^{\infty} \hat{u}\left(\frac{k}{L}\right) e^{i\frac{k}{L}x} \qquad \text{and} \qquad \hat{u}\left(\frac{k}{L}\right) = \frac{1}{2\pi}\int_{-L\pi}^{L\pi} u(x)\,e^{-i\frac{k}{L}x}\,dx \,.$$

We now call $\frac{k}{L} = \omega$ and let $L \to \infty$. In this limit, the sum above turns to an integral, and we get the Fourier transform formulas listed as the top case on Figure II4.0-1:

$$u(x) = \int_{-\infty}^{\infty} \hat{u}(\omega)\,e^{i\omega x}\,d\omega \qquad \text{and} \qquad \hat{u}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)\,e^{-i\omega x}\,dx \quad . (1)$$

Explained in words:

- when making the period wider, we needed to use an ever dense set of frequencies (no longer just just e^{ikx} with k integer, but increasingly many intermediate frequencies as well),

- as $L \to \infty$, a continuum of frequences are needed; instead of summing them to represent our function, we need to integrate over them all, and

- the domains for both integrals become [-∞,∞].

The Fourier transform operates on functions defined on [-∞,∞], and represents these as a superposition of a continuum of basic Fourier modes.

Convergence of a Fourier transform is assured if $\int_{-\infty}^{\infty} |u(x)|\,dx < \infty$. Like for a Fourier series, many important uses of the Fourier transform follow from differentiating or integrating (1). Table 1 summarizes a number of key properties of the Fourier transform. Table 2 gives some examples of the transforms of simple functions.

Some properties of the Fourier transform

General functions	Transform	Comment
$\frac{du}{dx}$	$i\,\omega\,\hat{u}(\omega)$	Turns space derivatives into algebraic factors; key to solving many ODEs and turning some PDEs into ODEs. Derivation: $\widehat{\left(\frac{du}{dx}\right)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{du}{dx}e^{-i\omega x}dx = \left[\frac{u\,e^{-i\omega x}}{2\pi}\right]_{-\infty}^{\infty} + \frac{i\omega}{2\pi}\int_{-\infty}^{\infty}u\,e^{-i\omega x}dx = i\,\omega\,\hat{u}$
$\frac{d^2u}{dx^2}$	$-\omega^2\,\hat{u}(\omega)$	Immediate from case above
$x\,u(x)$	$i\,\frac{d\hat{u}(\omega)}{d\omega}$	Can for ex. be used to obtain integral representations of variable coefficient ODEs - key to many asymptotic estimates. Ex: The Fourier transform to Airy's eq. $y'' - x\,y = 0$ satisfies $-\omega^2\hat{y} - i\,\hat{y}' = 0 \Rightarrow \hat{y} = A\,e^{i\omega^3/3};\ y(x) = \int_{-\infty}^{\infty}A\,e^{i\omega^3/3}\,e^{i\omega x}\,d\omega = A\int_{-\infty}^{\infty}\cos\left(\frac{\omega^3}{3}+\omega x\right)d\omega$
$u(x-\alpha)$	$e^{-i\omega\alpha}\,\hat{u}(\omega)$	Translation of a function introduces an exponential factor in the transform
$\frac{1}{2\pi}\int_{-\infty}^{\infty}f(\xi)g(x-\xi)d\xi$	$\hat{f}(\omega)\cdot\hat{g}(\omega)$	Convolution theorem: Large number of applications - some described in connection with the DFT - FFT. Derivation: Given $f(x)$, $g(x)$, which function $h(x)$ has the transform $\hat{h}(\omega) = \hat{f}(\omega)\cdot\hat{g}(\omega)$? $h(x) = \int_{-\infty}^{\infty}\hat{f}(\omega)\,\hat{g}(\omega)\,e^{i\omega x}d\omega = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}f(\xi)\,e^{-i\omega\xi}\,d\xi\right]\hat{g}(\omega)\,e^{i\omega x}\,d\omega =$ $= \frac{1}{2\pi}\int_{-\infty}^{\infty}f(\xi)\left[\int_{-\infty}^{\infty}\hat{g}(\omega)\,e^{i\omega(x-\xi)}\,d\omega\right]d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty}f(\xi)\,g(x-\xi)\,d\xi$
$\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert u(\xi)\rvert^2 d\xi = \int_{-\infty}^{\infty}\lvert\hat{u}(\omega)\rvert^2\,d\omega$		Parseval's relation: Setting $x = 0$ in the derivation of the Convolution theorem above gives $h(0) = \int_{-\infty}^{\infty}\hat{f}(\omega)\,\hat{g}(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}f(\xi)\,g(-\xi)\,d\xi$; Result follows from setting $f(x) = u(x),\ g(x) = \overline{u(-x)}$.

Table 1. Some properties of the Fourier transform.

There is unfortunately a considerable confusion between different texts about how to best define Fourier transforms. Our convention was stated in equation (1) - the function $u(x)$ is a straightforward superposition of the basic modes $e^{i\omega x}$. The transform using this choice, here denoted $\hat{u}_1(\omega)$, and two other versions are listed below:

Convention here:	$u(x) = \int_{-\infty}^{\infty}\hat{u}_1(\omega)\,e^{i\omega x}d\omega$;	$\hat{u}_1(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}u(x)\,e^{-i\omega x}dx$
Reversal of sign in exponent	$u(x) = \int_{-\infty}^{\infty}\hat{u}_2(\omega)\,e^{-i\omega x}d\omega$;	$\hat{u}_2(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}u(x)\,e^{i\omega x}dx$
Scaling of frequencies by factor 2π	$u(x) = \int_{-\infty}^{\infty}\hat{u}_3(\omega)\,e^{2\pi i\omega x}d\omega$;	$\hat{u}_3(\omega) = \int_{-\infty}^{\infty}u(x)\,e^{-2\pi i\omega x}dx$

Some examples of Fourier transforms

Functions	Transform	Comment
$\delta(x-x_0)$	$\frac{1}{2\pi} e^{-i\omega x_0}$	Transform of a delta function is a single trig. mode
$e^{i\omega_0 x}$	$\delta(\omega-\omega_0)$	Transform of a single trig. mode is a delta function
$e^{-\alpha x^2}$	$\frac{1}{\sqrt{4\pi\alpha}} e^{-\omega^2/(4\alpha)}$	Some functions have transforms which are simply scaled copies of themselves. The transform of a real even function is again real and even.
$\sqrt{\frac{\pi}{\alpha}}\, e^{-x^2/(4\alpha)}$	$e^{-\alpha\omega^2}$	Different scaling of case above
$\frac{1}{\cosh \alpha x}$	$\frac{1}{2\alpha \cosh\left(\frac{\pi\omega}{2\alpha}\right)}$	
$1/\sqrt{\lvert x\rvert}$	$\frac{1}{\sqrt{2\pi\lvert\omega\rvert}}$	Note: singular at the origin
$x e^{-\alpha x^2}$	$-\frac{i\omega}{4\alpha\sqrt{\pi\alpha}} e^{-\omega^2/(4\alpha)}$	Derivative of Gaussian - transform of a real odd function is again odd but purely imaginary
$\begin{cases} 0 & \lvert x\rvert > \alpha \\ 1 & \lvert x\rvert \le \alpha \end{cases}$	$\frac{1}{\pi} \frac{\sin \alpha\omega}{\omega}$	Transform for a step function decays for large ω only like $O(1/\omega)$.
$e^{-\alpha\lvert x\rvert}$	$\frac{\alpha}{\pi(\omega^2+\alpha^2)}$	If the derivative is discontinuous, the decay rate becomes $O(1/\omega^2)$ (like for a Fourier series)

Table 2. Some examples of Fourier transforms.

If we know a Fourier transform in one of these systems, we can quickly convert to any of the other by taking the actions given in the chart below:

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FROM	$\hat{u}_1$	$\hat{u}_2$	$\hat{u}_3$
$\hat{u}_1$	-	$\omega \to -\omega$	$\omega \to 2\pi\omega$ Multiply by 2π
$\hat{u}_2$	$\omega \to -\omega$	-	$\omega \to -2\pi\omega$ Multiply by 2π
$\hat{u}_3$	$\omega \to \omega/(2\pi)$ Divide by 2π	$\omega \to -\omega/(2\pi)$ Divide by 2π	-

As a single example of the very many uses of Fourier transforms, we consider again the problem of the 2-D heat equation - this time on an infinite interval:

Example: Solve the 1-D heat equation:

partial differential equation (PDE) $\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2}$ $(\sigma > 0)$

on the infinite interval (-∞,∞), subject to

initial condition (IC) $u(x,0) = f(x)$.

Solution: Following the idea in the example in Section II4.1, we view the IC as a superposition of individual Fourier modes

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\, d\omega$$

For each of these modes, we can immediately write down the time dependent solution to the PDE:

$$e^{-\sigma\omega^2 t}\, e^{i\omega x} \quad .$$

Hence, the complete solution (with the general IC; superposing the modes back together again) is

$$u(x,t) = \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-\sigma\omega^2 t}\, e^{i\omega x}\, d\omega \quad . \qquad (2)$$

In this case of infinite interval, we can proceed further (beyond what we could in the previous [0,1]- Fourier series case). When the explicit expression for $\hat{f}(\omega)$ is inserted, (2) becomes a double integral. This can be rearranged so that one becomes independent of *f*, and can be evaluated explicitly:

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi)\, e^{-i\omega\xi}\, d\xi \right] e^{-\sigma\omega^2 t}\, e^{i\omega x}\, d\omega = \\ &= \int_{-\infty}^{\infty} f(\xi) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\sigma\omega^2 t}\, e^{i\omega(x-\xi)}\, d\omega \right] d\xi = \\ &= \int_{-\infty}^{\infty} f(\xi)\; g(\xi - x, t)\; d\xi \, . \end{aligned}$$

The function $g(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\sigma\omega^2 t} e^{-i\omega z}\, d\omega$ is independent of *f*. A very easy way to evaluate $g(z,t)$ in closed form uses Cauchy's theorem for complex integration - unfortunately requiring background beyond the scope of this book. Omitting these steps

$$g(z,t) = \frac{1}{\sqrt{4\pi\sigma t}}\, e^{-z^2/(4\sigma t)} \quad ,$$

and hence

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi) \left[\frac{1}{\sqrt{4\pi\sigma t}}\, e^{-(\xi - x)^2/(4\sigma t)} \right] d\xi \quad .$$

The function inside the sqare bracket is known as the *influence function* - it describes in a quite direct way how the solution $u(x,t)$ at some later point in time and space is influenced by the IC $f(x)$.

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