4.3 Discrete Fourier transform (DFT)

For the Fourier transform (function u defined on $(-\infty,\infty)$), the basic modes present were

$$e^{i\omega x}\ , \qquad \omega \in R \qquad \text{(all real numbers).}$$

In the case of the Fourier series (function u periodic over $(-\pi,\pi)$), the modes present were restricted to

$$e^{ikx}\ , \qquad k \in Z \qquad \text{(the integers - positive and negative).}$$

For the discrete case (function u again 2π–periodic, but defined only at the N points $x_j = j \cdot \frac{2\pi}{N}$), the modes are limited even further:

$$e^{ikx}\ , \qquad k = 0, 1, \ldots, N-1\ .$$

Modes for which the k-values differ by N (or any integer multiple of N) become indistinguishable, as seen in Figure 1.

Fourier mode k takes at $x_j = j\frac{2\pi}{N}$ the values $e^{ikj\frac{2\pi}{N}}$, $j = 0,1,\ldots, N-1$. Mode $k + p\cdot N$ (p integer) takes similarly the values $e^{i(k+pN)j\frac{2\pi}{N}} = e^{ikj\frac{2\pi}{N}} \cdot e^{pNj\frac{2\pi}{N}} = e^{ikj\frac{2\pi}{N}}$, $j = 0,1,\ldots, N-1$, i.e. exactly the same values.

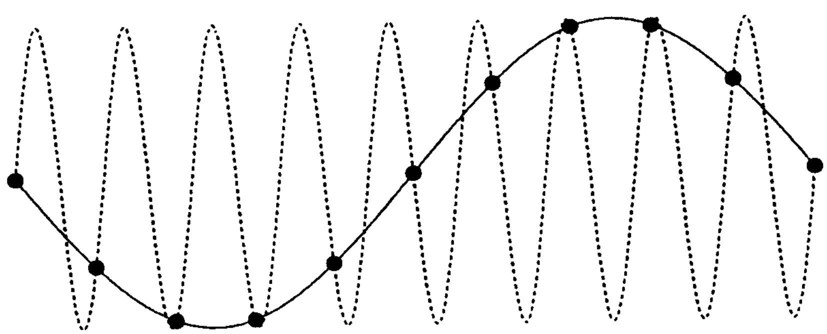

Figure 1. The functions $\sin x$ and $-\sin 9x$ are indistinguishable on the grid $x_j = 2\pi j / 10$, $j = 0, 1, \ldots, 10$.

If a (-π,π)-periodic function $u(x)$, with the Fourier series

$$u(x) = \sum_{k=-\infty}^{\infty} \hat{u}_{k\ (cont)}\, e^{ikx} \qquad ,$$

is sampled at the discrete points $x_j = j \cdot \frac{2\pi}{N}$, it can, at these locations, be represented as

$$u(x) = \sum_{k=0}^{N-1} \hat{u}_{k\ (discr)}\, e^{ikx} \qquad , \tag{1}$$

where

$$\hat{u}_{k\ (discr)} = \sum_{p=-\infty}^{\infty} \hat{u}_{k+pN\ (cont)} \quad . \tag{2}$$

It is immaterial if the sum in (1) extends from 0 to N-1 or $-\frac{N}{2}+1$ to $\frac{N}{2}$ (assuming N is even - or indeed over any range with N consecutive integers). The latter range will turn out to be the natural one in many cases.

Both the input data and the output result for the DFT consists of N complex numbers. It is more natural to index both of these sets by the integers 0, 1, ... , N-1 than to, in one of the cases, consider equi-spaced points on (-π,π). Hence, we alter the physical interval to (0,N), and obtain the last case illustrated in Figure II4.0-1. The transform pair we obtain

$$u_j = \sum_{k=0}^{N-1} \hat{u}_k\, e^{2\pi i k j/N} \qquad , \qquad j = 0, 1, \ldots, N-1 \qquad , \tag{3}$$

and

$$\hat{u}_k = \frac{1}{N} \sum_{k=0}^{N-1} u_j\, e^{-2\pi i k j/N} \qquad , \qquad k = 0, 1, \ldots, N-1 \tag{4}$$

is often best expressed in matrix form. In case of (3), this becomes

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2N-2} \\ \vdots & \vdots & & & & \\ \vdots & \vdots & & & & \\ 1 & \omega^{N-1} & \cdots & \cdots & \cdots & \omega^{(N-1)^2} \end{bmatrix} \begin{bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \vdots \\ \hat{u}_{N-1} \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{N-1} \end{bmatrix} \tag{5}$$

where $\omega = e^{2\pi i/N}$. Similarly for the inverse transform (4):

$$\frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \cdots & \omega^{-N+1} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \cdots & \omega^{-2N+2} \\ \vdots & \vdots & & & & \\ \vdots & \vdots & & & & \\ 1 & \omega^{-N+1} & \cdots & \cdots & \cdots & \omega^{-(N-1)^2} \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \vdots \\ \hat{u}_{N-1} \end{bmatrix} \tag{6}$$

We can immediately verify that the product of the two matrices above is N times the identity matrix (each element of the product is the sum of a finite geometric progression). This confirms that (3), (4) indeed constitutes a transform and its inverse.

In the Section III5, we find that the fast Fourier transform (FFT) amounts to exploiting that the matrix in (5) (which is entirely independent of the data) can be factored in a product of very sparse matrices. The matrix × vector product in (5) can then then instead be performed by multiplying the vector in turn with these sparse matrices.

Discrete convolution theorem

One of the most important applications of the FFT algorithm is that it allows periodic discrete convolutions to be calculated rapidly. If three vectors

$$[\, x_0 \,, x_1, \,\ldots\, , x_{N\text{-}1}] \;,\; [\, y_0 \,, y_1, \,\ldots\, , y_{N\text{-}1}] \;,\; [\, z_0 \,, z_1, \,\ldots\, , z_{N\text{-}1}] \tag{7}$$

satisfy

$$\begin{bmatrix} z_0 & z_{N-1} & z_{N-2} & \ddots & z_1 \\ z_1 & z_0 & z_{N-1} & \ddots & z_2 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ z_{N-1} & z_{N-2} & \ddots & z_1 & z_0 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ \vdots \\ x_{N-1} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ \vdots \\ y_{N-1} \end{bmatrix} \quad , \tag{8}$$

then their DFT coefficients satisfy

$$\begin{bmatrix} \hat{z}_0 & & & & \\ & \hat{z}_1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \hat{z}_{N-1} \end{bmatrix}\begin{bmatrix} \hat{x}_0 \\ \hat{x}_1 \\ \vdots \\ \vdots \\ \hat{x}_{N-1} \end{bmatrix} = \frac{1}{N}\begin{bmatrix} \hat{y}_0 \\ \hat{y}_1 \\ \vdots \\ \vdots \\ \hat{y}_{N-1} \end{bmatrix} \quad . \tag{9}$$

Together with the FFT algorithm, this result offers a very fast way to calculate any one of the vectors in (7) if the other two are provided.

The following illustration of how the discrete convolution theorem can be derived in the case of N = 4 generalizes immediately to any size. We multiply (9) from the left by the matrix in (5) and in-between the matrix and vector in (9), we insert the matrices from (6) and (5):

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{bmatrix} \begin{bmatrix} \hat{z}_0 & & & \\ & \hat{z}_1 & & \\ & & \hat{z}_2 & \\ & & & \hat{z}_3 \end{bmatrix} \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{bmatrix} \begin{bmatrix} \hat{x}_0 \\ \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{bmatrix} \begin{bmatrix} \hat{y}_0 \\ \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \end{bmatrix}$$

We now cancel the $1/N$, multiply together the three leftmost matrices, and apply (5) to the two matrix×vector products:

$$\begin{bmatrix} \hat{z}_0+\hat{z}_1+\hat{z}_2+\hat{z}_3 & \hat{z}_0+\hat{z}_1\omega^3+\hat{z}_2\omega^6+\hat{z}_3\omega^9 & \hat{z}_0+\hat{z}_1\omega^2+\hat{z}_2\omega^4+\hat{z}_3\omega^6 & \hat{z}_0+\hat{z}_1\omega+\hat{z}_2\omega^2+\hat{z}_3\omega^3 \\ \hat{z}_0+\hat{z}_1\omega+\hat{z}_2\omega^2+\hat{z}_3\omega^3 & \hat{z}_0+\hat{z}_1+\hat{z}_2+\hat{z}_3 & \hat{z}_0+\hat{z}_1\omega^3+\hat{z}_2\omega^6+\hat{z}_3\omega^9 & \hat{z}_0+\hat{z}_1\omega^2+\hat{z}_2\omega^4+\hat{z}_3\omega^6 \\ \hat{z}_0+\hat{z}_1\omega^2+\hat{z}_2\omega^4+\hat{z}_3\omega^6 & \hat{z}_0+\hat{z}_1\omega+\hat{z}_2\omega^2+\hat{z}_3\omega^3 & \hat{z}_0+\hat{z}_1+\hat{z}_2+\hat{z}_3 & \hat{z}_0+\hat{z}_1\omega^3+\hat{z}_2\omega^6+\hat{z}_3\omega^9 \\ \hat{z}_0+\hat{z}_1\omega^3+\hat{z}_2\omega^6+\hat{z}_3\omega^9 & \hat{z}_0+\hat{z}_1\omega^2+\hat{z}_2\omega^4+\hat{z}_3\omega^6 & \hat{z}_0+\hat{z}_1\omega+\hat{z}_2\omega^2+\hat{z}_3\omega^3 & \hat{z}_0+\hat{z}_1+\hat{z}_2+\hat{z}_3 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

From (5) follows that the elements in the matrix above agree with the ones in the matrix in (8).

Numbering of modes / trigonometric interpolation

Let us assume that N is odd (not critical - but slightly simplifies the algebra). Suppose we are given u_0 , u_1 , ... , $u_{N\text{-}1}$, all real numbers, and that we have calculated complex DFT coefficients $\hat{u}_0$, $\hat{u}_1$, ... , $\hat{u}_{N-1}$. Then

$$u_j = \sum_{k=0}^{N-1} \hat{u}_k \, e^{2\pi i k j/N} \ , \ j = 0, 1, \ldots, N\text{-}1 \ .$$

An interesting idea now is to now evaluate the sum above also for j not an integer - we would expect to get an interpolated result for $u(x)$ in-between the original grid points. This leads to a nasty surprise: Even when all the original data is real, the interpolant becomes complex in-between the node points (j = 0,1,...,N-1).

To resolve this, remember that our Fourier modes were

$$\hat{u}_0 \, , \, \hat{u}_1 \, , \, \hat{u}_2 \, , \, \ldots \, , \, \hat{u}_{\frac{N-1}{2}} \, , \, \hat{u}_{\frac{N+1}{2}} \, , \, \ldots \, , \, \hat{u}_{N-2} \, , \, \hat{u}_{N-1}$$

corresponding to wave numbers $\quad 0 \, , \, 1 \, , \, 2 \, , \, \ldots \, , \frac{N-1}{2} \, , \, \frac{N+1}{2} \, , \, \ldots \, , \, N\text{-}2 \, , \, N\text{-}1 \, .$

can view as modes $\quad 0 \, , \, 1 \, , \, 2 \, , \, \ldots \, , \frac{N-1}{2} \, , -\frac{N-1}{2} \, , \, \ldots \, , \, \text{-}2 \, , \, \text{-}1 \, .$

Interpolation requires the modes to be interpreted this way.

We next verify that real data indeed gives a real interpolant when mode numbers are numbered this way. Suppose u_j, j = 0, 1, ... , N-1 real. A DFT gives us the coefficients

$$\hat{u}_k = \frac{1}{N} \sum_{j=0}^{N-1} u_j \, e^{-2\pi i k j/N}$$

Complex conjugating this equation will, for the right hand side, have the same effect as if k was changed for -k, i.e.:

$$\hat{u}_k = \overline{\hat{u}_{-k}} \qquad \text{for all } k \quad \text{(in particular, } \hat{u}_0 \text{ is real).}$$

Now, interpolate (evaluate u_j) with j not an integer (using our new mode numbering):

$$\overline{u_j} = \overline{\sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \hat{u}_k \, e^{2\pi i k j/N}} = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \hat{u}_{-k} \, e^{2\pi i (-k) j/N} = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \hat{u}_k \, e^{2\pi i k j/N} = u_j \; .$$

Therefore, the result is real. Interpolation (differentiation etc.) of trigonometric polynomials work now as expexted.

If N had been even, there would have been an additional mode $\pm\frac{N}{2}$ present - known as the 'reflection'- or Nyquist frequency. In most cases, we can set its coefficient to zero (or ignore its presence).

Character of transform for different cases of input data

If the N input numbers to a DFT are complex, so are the output ones. However, we noted in equation (II1-6) that the complex Fourier coefficients of a real function appeared in complex-conjugated pairs. A similar result holds for the Fourier transform - and again for the DFT. Table 1. This is seen as the second case in the table, labeled "Real". It again illustrates how it is is necessary to interpret the modes in the order [0, 1, 2, ..., -2, -1] just described. In this table, N is assumed to be even - by dar the common case in computations (due to the FFT algorithm, which is particularly simple and effective for sizes that are powers of two). The mode N/2 then does not quite 'fit the pattern'. This is in practice of little consequence; as noted above, it can be safely ignored in virtually all cases where the DFT is used to approximate a Fourier series or a Fourier transform.

Transform type	Structure of Input	Structure of Output	Name of code included
Complex	General, complex Re: $a_0\ a_1\ \ldots\ \ldots\ a_{\frac{N}{2}-1}\ a_{\frac{N}{2}}\ a_{\frac{N}{2}+1}\ \ldots\ \ldots\ a_{N-1}$ Im: $b_0\ b_1\ \ldots\ \ldots\ b_{\frac{N}{2}-1}\ b_{\frac{N}{2}}\ b_{\frac{N}{2}+1}\ \ldots\ \ldots\ b_{N-1}$	General, complex Re: $c_0\ c_1\ \ldots\ \ldots\ c_{\frac{N}{2}-1}\ c_{\frac{N}{2}}\ c_{\frac{N}{2}+1}\ \ldots\ \ldots\ c_{N-1}$ Im: $d_0\ d_1\ \ldots\ \ldots\ d_{\frac{N}{2}-1}\ d_{\frac{N}{2}}\ d_{\frac{N}{2}+1}\ \ldots\ \ldots\ d_{N-1}$	**FFT**
Real	Real Re: $a_0\ a_1\ \ldots\ \ldots\ a_{\frac{N}{2}-1}\ a_{\frac{N}{2}}\ a_{\frac{N}{2}+1}\ \ldots\ \ldots\ a_{N-1}$ Im: $0\ 0\ \ldots\ \ldots\ 0\ 0\ 0\ \ldots\ \ldots\ 0$	Real part symmetric, imaginary part anti-symmetric Re: $c_0\ c_1\ \ldots\ \ldots\ c_{\frac{N}{2}-1}\ c_{\frac{N}{2}}\ c_{\frac{N}{2}-1}\ \ldots\ \ldots\ c_1$ Im: $0\ d_1\ \ldots\ \ldots\ d_{\frac{N}{2}-1}\ 0\ -d_{\frac{N}{2}-1}\ \ldots\ \ldots\ -d_1$	**-**
Cosine	Real , symmetric Re: $a_0\ a_1\ \ldots\ \ldots\ a_{\frac{N}{2}-1}\ a_{\frac{N}{2}}\ a_{\frac{N}{2}-1}\ \ldots\ \ldots\ a_1$ Im: $0\ 0\ \ldots\ \ldots\ 0\ 0\ 0\ \ldots\ \ldots\ 0$	Real, symmetric Re: $c_0\ c_1\ \ldots\ \ldots\ c_{\frac{N}{2}-1}\ c_{\frac{N}{2}}\ c_{\frac{N}{2}-1}\ \ldots\ \ldots\ c_1$ Im: $0\ 0\ \ldots\ \ldots\ 0\ 0\ 0\ \ldots\ \ldots\ 0$	**FCT**
Sine	Real, anti-symmetric Re: $0\ a_1\ \ldots\ \ldots\ a_{\frac{N}{2}-1}\ 0\ -a_{\frac{N}{2}-1}\ \ldots\ \ldots\ -a_1$ Im: $0\ 0\ \ldots\ \ldots\ 0\ 0\ 0\ \ldots\ \ldots\ 0$	Imaginary, anti-symmetric Re: $0\ 0\ \ldots\ \ldots\ 0\ 0\ 0\ \ldots\ \ldots\ 0$ Im: $0\ d_1\ \ldots\ \ldots\ d_{\frac{N}{2}-1}\ 0\ -d_{\frac{N}{2}-1}\ \ldots\ \ldots\ -d_1$	**FST**

Table 1. Comparison between a general complex DFT and some special cases of it. The heavy lines surround the data vectors which the different routines require as input and provide as output. *N* is assumed to be even.