4.4 2-D Fourier transform

We can graphically display the paths (both ways) between $u(x)$ and its 1-D Fourier transform $\hat{u}(\omega)$ as

$$\begin{tikzcd}[column sep=huge]
\boxed{u(x)} \arrow[r, bend left, "\frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)\, e^{-i\omega x}\, dx"] & \boxed{\hat{u}(\omega)} \arrow[l, bend left, "\int_{-\infty}^{\infty} \hat{u}(x)\, e^{i\omega x}\, dx"]
\end{tikzcd}$$

In 2-D, the Fourier transform pair is defined as

$$u(x_1, x_2) \quad = \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{u}(\omega_1, \omega_2)\, e^{\,i(\omega_1 x_1 + \omega_2 x_2)}\, d\omega_1\, d\omega_2 \tag{1}$$

where

$$\hat{u}(\omega_1, \omega_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x_1, x_2)\; e^{-i(\omega_1 x_1 + \omega_2 x_2)}\, dx_1\, dx_2 \quad . \tag{2}$$

This can be graphically displayed as a 2-step process using only 1-D Fourier transforms:

$$\begin{tikzcd}[column sep=huge]
\boxed{u(x_1, x_2)} \arrow[r, bend left, "\frac{1}{2\pi}\int_{-\infty}^{\infty} u(x_1, x_2)\, e^{-i\omega_1 x_1}\, dx_1"] & \boxed{\check{u}(\omega_1, x_2)} \arrow[l, bend left, "\int_{-\infty}^{\infty} \check{u}(\omega_1, x_2)\, e^{i\omega_1 x_1}\, d\omega_1"] \arrow[r, bend left, "\frac{1}{2\pi}\int_{-\infty}^{\infty} \check{u}(\omega_1, x_2)\, e^{-i\omega_2 x_2}\, dx_2"] & \boxed{\hat{u}(\omega_1, \omega_2)} \arrow[l, bend left, "\int_{-\infty}^{\infty} \hat{u}(\omega_1, \omega_2)\, e^{i\omega_2 x_2}\, d\omega_2"]
\end{tikzcd}$$

The path left-to-right clearly constitutes (2), and the return path (1). We can note:

- it was of no consequence that we here transformed in the x_1-direction first and x_2 second; reversed order gives the same result,
- that equations (1) and (2) form a transform pair follows from this property of 1-D transforms,
- 2-D transforms are most easily performed as successive 1-D ones. In particular, for the DFT case, 1-D FFTs applied on first rows, and then columns (or vice versa) provides 2-D transforms - there is no need for special 2-D (or higher-D codes).

One of the most noteworthy (and, for the FT method for tomography, critically important) properties of the 2-D Fourier transform is the following:

If a function is rotated around the origin in (x_1, x_2)- space, its Fourier transform becomes rotated by the same angle in (ω_1, ω_2)- space.

If the original function $u(x_1,x_2)$ has the Fourier transform

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x_1,x_2)\, e^{-i(\omega_1 x_1+\omega_2 x_2)}\, dx_1\, dx_2 = \hat{u}(\omega_1,\omega_2)\ ,$$

then the rotated function $u(x_1\cos\theta + x_2\sin\theta, -x_1\sin\theta + x_2\cos\theta)$ has the transform

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x_1\cos\theta + x_2\sin\theta, -x_1\sin\theta + x_2\cos\theta)\, e^{-i(\omega_1 x_1+\omega_2 x_2)}\, dx_1\, dx_2 =$$

call $\left\{\begin{array}{l} x_1' = x_1\cos\theta + x_2\sin\theta \\ x_2' = -x_1\sin\theta + x_2\cos\theta \end{array}\right\}$, then $\left\{\begin{array}{l} x_1 = x_1'\cos\theta - x_2'\sin\theta \\ x_2 = x_1'\sin\theta + x_2'\cos\theta \end{array}\right\}$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x_1',x_2')\, e^{-i[\omega_1(x_1'\cos\theta - x_2'\sin\theta)+\omega_2 x_1'\sin\theta + x_2'\cos\theta)]}\, dx_1\, dx_2 =$$

reorder terms in exponent; change integration variables from x_1, x_2 to x_1', x_2' (no extra factor since Jacobian = 1)

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x_1',x_2')\, e^{-i[(\omega_1\cos\theta - \omega_2\sin\theta)x_1' + (-\omega_1\sin\theta + \omega_2\cos\theta)x_2']}\, dx_1'\, dx_2' =$$

irrelevant that what dummy integration variables called, so

$$= \hat{u}(\omega_1\cos\theta + \omega_2\sin\theta,\ -\omega_1\sin\theta + \omega_2\cos\theta)\ .$$

Hence, the Fourier transform has been rotated the same angle around the origin in ω_1, ω_2- space as the function was in x_1, x_2- space.

The result generalizes immediately to 3-D - of significance for the display of volume data. For an easy proof in the 3-D case, it suffices to note that any rotation in 3-space can be carried out as successive rotations around the three axes. For each of these cases, the proof above applies.