4.1 Wave Function

1-D wave function:

A progressive wave may at one instance in time look like what is shown in Figure 1:

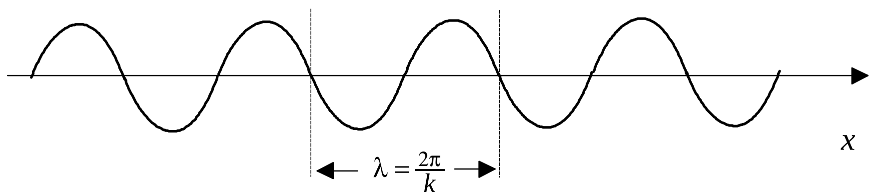

Figure 1. Snapshot of a progressive sinusoidal wave.

The variable that is displayed here need not correspond to any sideways deflections - for sound waves, it may be pressure; for light, it may be the strength of an electric or magnetic field. We consider here only the case in which it translates at a fixed speed as a function of time. The simplest such a progressive wave can be written

$$\phi\,(x, t) = \phi_o \; \cos\,(k\,x - \omega\,t) \quad . \tag{1}$$

We denote	*wavenumber*	k			
	wave length	λ	=	$2\pi / k$	,
	frequency	v	=	$\omega\ / (2\pi)$	.

If t is increased by one and x by ω/k , the argument in the sine function is unchanged, Hence, the wave in (1) travels with

$$(\textit{phase})\ \textit{speed} \qquad c_{(p)} \quad = \quad \omega/k \qquad .$$

(We will later come across another speed, *group speed*, c_g ; if only *speed* is mentioned or no subscript is given, c_p is assumed).

For almost all waves, ω is not a constant but a function of k ; $\omega = \omega(k)$. Often, this relation takes the form $\omega = c \cdot k$. In that case, the speed of the wave is c and independent of the wavenumber. This is the case for example for sound and for light (in vacuum). In other cases, the wave speed depends on k. One such case is waves on deep water. As we will show in Section 4.3, we then have $\omega = \sqrt{g \cdot k}$ and consequently $c_p = \sqrt{g/k}$ where g denotes the acceleration of gravity (in this case, we have to be careful to specify c_p since there is a separate group speed c_g = ½ c_p ; more about that later).

It is often very convenient to write (1) as

$$\phi(x, t) = \phi_o \quad Re \quad e^{i(kx - \omega t)} \quad . \qquad (2)$$

This follows from Euler's equation $e^{ix} = \cos x + i \sin x$ where i is the imaginary unit, satisfying $i^2 = -1$. Euler's equation is most easily verified by Taylor expansion: $e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} --++... =$
$(1 - \frac{x^2}{2!} + \frac{x^4}{4!} -+...) + i\,(x - \frac{x^3}{3!} + \frac{x^5}{5!} -+...) =$
$= \cos x + i \sin x$.

A lot of trigonometric manipulations are very much easier if one uses the complex exponential function rather than manipulate sines and cosines directly. The latter would this often require the use of a large number of trigonometric identities. Such identities are often almost immediate (and indeed easiest to derive)with Euler's formula.

Some illustrative examples of this are:

Addition theorem: **$\cos(\alpha+\beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$, $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$:**

$e^{i(\alpha+\beta)} = e^{i\alpha}\, e^{i\beta} \Rightarrow \cos(\alpha+\beta) + i\sin(\alpha+\beta) = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$; multiply out the RHS, the two formulas follow by considering the real and imaginary parts,

Multiple angle formula: **$\cos(3\alpha) = 4\cos^3\alpha - 3\cos\alpha$, $\sin(3\alpha) = 3\sin\alpha - 4\sin^3\alpha$:**

$e^{3i\alpha} = (e^{i\alpha})^3$; express each exponential through Euler's formula, expand out the RHS and agin separate real and imaginary parts.

Trigonometric sum: **$\sin x + \sin 2x + \sin 3x + ... + \sin Nx = (\cos ½ x - \cos (N+½)x)/(2 \sin ½ x)$.**

$\Sigma = Im\, \Sigma_{k=1}^{N} e^{ikx} = Im \frac{e^{i(N+1)x} - e^{ix}}{e^{ix} - 1}$ (since sum was converted to a geometric progression); multiply up and down with $e^{-ix/2}$ and pick out imaginary part of the result.

Interpret amplitude modulation cos εt of a carrier wave cos ωt as frequency modulations:

$(\cos \varepsilon t)(\cos \omega t) = ½\,(e^{i\varepsilon t} + e^{-i\varepsilon t}) \cdot ½\,(e^{i\omega t} + e^{-i\omega t}) = ¼\,(e^{i(\omega+\varepsilon)t} + e^{-i(\omega-\varepsilon)t} + e^{i(\omega-\varepsilon)t} + e^{-(\omega+\varepsilon)t}) = ½(\cos(\omega+\varepsilon)t + \cos(\omega-\varepsilon)t)$.

The message contained in this is that it is very much simpler algrbraically to deal with complex exponentials than with sines and cosines. Equation (2) is therefore usually preferrable over (1). It is even more convenient not to write "*Re*" all the time - we therefore write the 1-D wave function as

$$\phi(x, t) = \phi_o \quad e^{i(kx - \omega t)} \quad . \qquad (3)$$

with the understanding that we should take the imaginary part (or real part) when we want to relate it to something real and physical.

Not all wave forms are sinusoidal. However, by Fourier analysis (cf. Chapter ....), any other shape can be viewed as a linear superposition of sinusoidal waves of different frequencies k. Together with knowledge of the function $\omega = \omega(k)$, we can analyze how any initial wave form evolves in time.

<u>2-D wave function:</u>

The 2-D counterpart to (3) is

$$\phi\,(\underline{x}, t) = \phi_o \quad e^{\,i\,(\underline{k}\cdot\underline{x} - \omega\, t)} \tag{4}$$

where $\underline{x} = (x_1, x_2)$ and $\underline{k} = (k_1, k_2)$ are 2-component vectors.

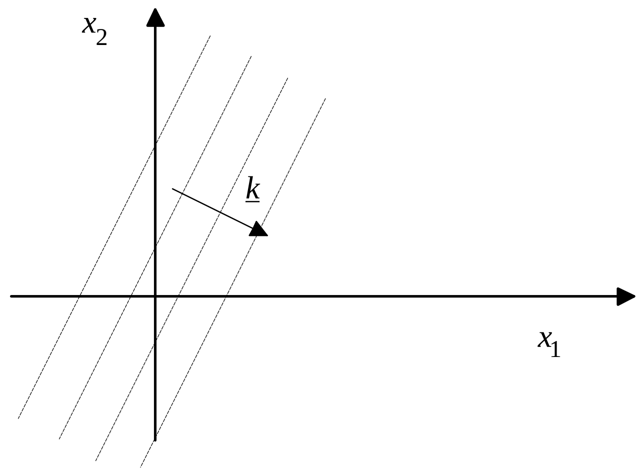

Figure 2. 2-D progressing wave. Wave crests marked with dotted lines - waves progress in direction of k - vector.

The wave $\phi\,(\underline{x}, t)$ given by (4) clearly reduces to (3) in case we introduce a (scalar) x - direction parallel to the $\underline{k}$ - vector. We can also note that it is unchanged if $\underline{x}$ moves along any direction orthogonal to $\underline{k}$. From the first observation follows that $\lambda = 2\pi / |\,\underline{k}\,|$ and $c_p = \omega / |\,\underline{k}\,|$.