4.2 Derivations of Equations for Waves along a String

Two kinds of waves will travel along a string under tension - transverse and longitudinal. Their speeds are typically vastly different. Transverse (sideways) oscillations are usually fairly slow, and visible, whereas longitudinal (lengthwise) waves would travel with the speed of sound in the material, maybe of the order of 1 km/s, while causing no visible deflections. In a loosely stretched 'slinky', both wave types can be seen traveling at about 10 m/s. The transversal waves in a string is the simplest case to obtain an equation for, and we will do that in subsection 4.2.1, followed by the longitudinal case in subsection 4.2.2.

4.1.1 Transverse Waves in 1-D

A string, with density ρ per unit length, is stretched in the x-direction with a tension force T (cf. Figure 1).

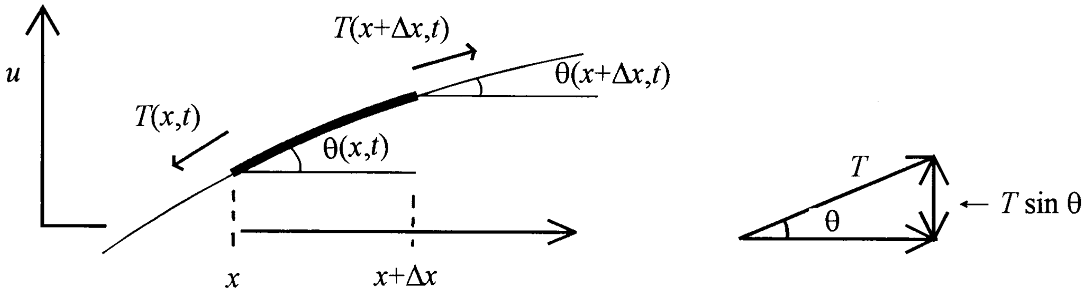

Figure 1. Illustration of infinitesimal section of transversally vibrating string.

At any time, the vertical forces on the small string segment must balance. They are

$$\underbrace{\rho \cdot \Delta x} \cdot \frac{\partial^2 u}{\partial t^2} = T(x+\Delta x, t)\,\sin\theta(x+\Delta x, t) - T(x, t)\,\sin\theta(x, t)$$

mass acce- leration difference between vertical tension forces at two ends

Assuming the deflection angles $\theta(x,t)$ are sufficiently small that $\sin\theta \approx \tan\theta = \frac{\partial u}{\partial x}$, we get after dividing both sides by Δx , and letting $\Delta x \to 0$

$$\rho(x)\,\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial x}\left(T\,\frac{\partial u}{\partial x}\right) \qquad .$$

Again, if the deflection is small, the tension T is approximately constant, and can be factored out. Calling $T/\rho = c^2$, we get the 1-D wave equation in its standard form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

4.2.2 Longitudinal waves in 1-D

In this case, we think of a rod instead of a string - it then resists both tension and compression. It is again useful to look at a small section of the rod (Figure 2):

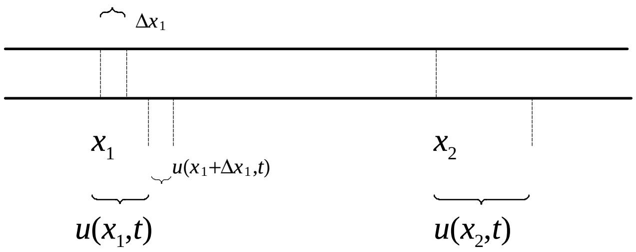

Figure 2. Section of rod as it undergoes longitudinal motion.

Two key quantities are

stress - force attempting to elongate (or compress) a rod section, and
strain - $\varepsilon = \dfrac{\text{elongation}}{\text{unstrained length}}$.

These two quantities are assumed related by Hooke's law

$$\text{stress} = E \cdot \text{strain}$$

where E is called the *modulus of elasticity*. The local strain at x_1 is $\varepsilon = \frac{u(x_1+\Delta x_1,t) - u(x_1,t)}{\Delta x_1}$; letting $\Delta x_1 \to 0$ gives

$$\varepsilon = \frac{\partial u}{\partial x} \quad .$$

Balancing the forces on the sample that used to be between x_1 and x_2 gives

$$F_x = \left\{ \begin{matrix} \text{cross-sectional} \\ \text{area } A \end{matrix} \right\} \cdot \left\{ \text{stress}\big|_{x_2,t} - \text{stress}\big|_{x_1,t} \right\} = A \cdot E \cdot \left\{ \text{strain}\big|_{x_2,t} - \text{strain}\big|_{x_1,t} \right\} =$$

$$= A \cdot E \cdot \left[\frac{\partial u}{\partial x}\Big|_{x_2,t} - \frac{\partial u}{\partial x}\Big|_{x_1,t} \right].$$

Newton's second law gives

$$F_x = \text{mass} \cdot \text{accel.} = \rho A (x_2 - x_1) \frac{\partial^2 u}{\partial \ {}^2} .$$

Putting the forces equal, and letting $x_1 \to x_2$ gives

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2} .$$

We have obtained the same wave equation as in the transverse case - only difference being that the speed now is $c_{\text{long}} = \sqrt{\frac{E}{\rho}}$ as opposed to $c_{\text{trans}} = \sqrt{\frac{T}{\rho}}$.