

## 4.3 Water Waves

In nature, one encounters a great variety of different wave phenomena in bodies of water. Table 1 summarizes a number of these, which can be found in lakes and oceans.

TABLE 1. *Types of wave phenomena in oceans / lakes*

Wave type	Cause	Physical mechanism	Period	Velocity	Region influenced
Sound	Sea life, ships	Compressibility	$10^{-1} - 10^{-5}$ s	1.52 km/s	Interior
Capillary ripples	Wind	Surface tension	$< 10^{-1}$ s	25-50 cm/s	Surface
Wind waves and swell	Wind	Gravity	1-25 s	2-40 m/s	Surface
Sieches	Earthquakes, storms	Gravity, resonance	minutes to hours	standing waves	Interior and surface of large lakes
Storm surges	Low pressure areas	Gravity and earth rotation	1 - 10 h	$\sim 100$ m/s	Coastal
Tsunami	Earthquakes, slides	Gravity	10 min - 2 h	$< 800$ km/h	Interior; on surface at shores
Internal waves	Stratification instabilities, tides	Gravity and density stratifications	2 min - 10 h	$< 5$ m/s	Layer of sharp density change
Tides	Moon and sun	External gravity fields	12 - 24 h	1700 km/h; bores a few km/h	Interior; on surface at shores (bores)
Planetary waves	Earth rotation	Gravity	$\sim 100$ days	1-10 km/h	Interior

We will here describe only the visually most obvious one - surface waves (third category in the list). To even get started, we will need to make a number of simplifying assumptions:

- no surface tension or wind effects,
- infinitely deep water,
- very small wave amplitude  $a$ ,
- consider only steady translating waves.

For waves on a string, we could write down a PDE which can be used to describe how any initial state evolves forward in time. For gravity waves, there is no single PDE that describes a general evolution of a surface disturbance.

### 4.3.1 Dispersion relation for deep water.

To make progress, we need to make still one more simplifying assumption, namely that the surface elevation  $\eta(x,t)$  for small amplitude  $a$  becomes approximately sinusoidal

$$\eta(x,t) = a \cos(kx - \omega t) \quad . \quad (1)$$

We outline below how one can show that  $k$  and  $\omega = \omega(k)$  become related through the *dispersion relation*

$$\omega^2 = g k \quad . \quad (2)$$

Here  $g$  stands for the acceleration of gravity;  $g \approx 9.8 \text{ m/s}^2$ . Denoting the wave length by  $\lambda = \frac{2\pi}{k}$ , the velocity of this wave becomes

$$c_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}} \quad ; \quad (3)$$

the time period  $T$  becomes  $T = 2\pi/\omega$  (the "p" in  $c_p$  indicates phase velocity - we will later come across another velocity  $c_g =$  group velocity). The fact that  $c_p$  grows proportionally with  $\sqrt{\lambda}$  leads to many notable features of water waves. Some of these are mentioned in Section .....

The following gives a flavor of the series of approximate arguments that are needed to obtain the key result (2):

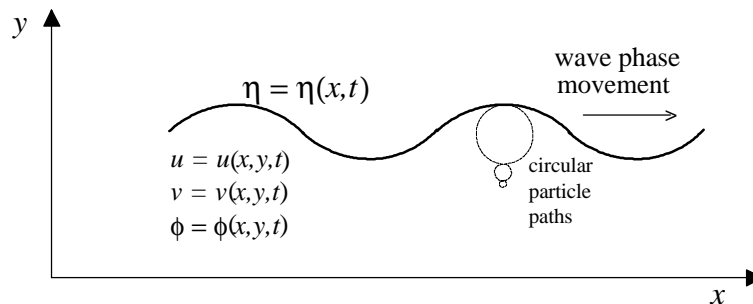


Figure 1. Some relevant notation for derivation of dispersion relation.

Basic equations and derivation of dispersion relation:

With the notation seen in Figure 2, the key equations become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (4) \quad \text{This expresses that the fluid is incompressible; inflow and outflow must balance for each infinitesimal 'box' of fluid,}$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad (5) \quad \text{The quantity } \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \text{ is called } \textit{vorticity}. \text{ In flows with no } \textit{viscosity}, \omega \text{ can be shown to remain unchanged in time for every fluid element. Clearly, } \omega = 0 \text{ holds for a stationary fluid; when waves are generated, the viscosity of water is much too low to be able to introduce any significant vorticity.}$$

$$\frac{\partial \eta}{\partial t} = v - u \frac{\partial \eta}{\partial x} \quad (6) \quad \text{A fluid element on the surface will remain on the surface; the rate of change of the surface elevation is due to}$$

- i. the vertical component of the fluid velocity, and
- ii. (the horizontal component)  $\cdot$  (the surface slope)

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2 + v^2) + g\eta = 0 \quad (7) \quad \text{One can show that it is possible to introduce a } \textit{velocity potential} \phi \text{ such that } \frac{\partial \phi}{\partial y} = v, \frac{\partial \phi}{\partial x} = u \text{ (the necessary property } \phi_{,xy} = \phi_{,yx} \text{ is satisfied because of (5)). About half a page of algebraic manipulation of the fundamental } \textit{Euler equations} \text{ for incompressible inviscid flow lead to (7) (when noting that the pressure is the same at every point on the surface), see for ex. Acheson (1990). We omit the details here.}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad (8) \quad \text{An immediate consequence of (4) and of the definition of } \phi.$$

The functions  $u, v, \phi$  all depend on  $x, y, t$  whereas the surface elevation  $\eta$  depends on  $x, t$  only. The equations (4), (5), (8) are valid throughout the fluid; (6), (7) only on the surface.

These equations above contain no assumptions of low amplitude; the combination (6)-(8) can be used for numerical solution of progressing (time varying or steady) waves. As these equations stand, they involve  $t$  and two space variables  $x$  and  $y$ . The technique of *conformal mapping* allows  $y$  to be eliminated, leading to highly effective numerical approaches (in  $x$ - and  $t$  only, but these are no longer expressible in the form of a differential equation). We will follow another path here, striving instead to motivate (2) in the limit of small amplitude.

The basic idea for simplifications is to note that when  $a$  is very small (compared to, say  $\lambda$ ), certain terms in the equations above become negligible. In (6),  $\frac{\partial \eta}{\partial t}$ ,  $v$ ,  $u$ , and  $\frac{\partial \eta}{\partial x}$  all become small roughly at the same rate as  $a$ . Hence, we ignore the product term, and thus simplify (6) to

$$\frac{\partial \eta}{\partial t} = v \quad \left( = \frac{\partial \phi}{\partial y} \right) \quad (9)$$

Equation (6) is valid on the surface  $y = \eta$ . We can change  $y = \eta$  to the simpler flat surface  $y = 0$  by again neglecting 'higher order' product terms:  $v(x, \eta, t) = v(x, 0, t) + \eta \frac{\partial v}{\partial y}(x, 0, t) + \dots$ . Similar treatment of (5) suggests that we on  $y = 0$  use

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad (10)$$

With  $\eta(x, t)$  of the (approximate) form (1), equation (9) tells that

$$\frac{\partial \phi}{\partial y} = \omega a \sin(kx - \omega t)$$

should hold on  $y = 0$ . This function  $\phi(x, y)$  is defined not only on the surface but also for all  $y < 0$ . It satisfies (8) and tends to zero as  $y \rightarrow -\infty$ . Inspection (or an easy derivation) reveals that

$$\phi = \frac{a\omega}{k} e^{ky} \sin(kx - \omega t) \quad (11)$$

(uniquely) satisfies all the requirements. Substituting (1) and (11) into (10) (valid at the surface  $y = 0$ ) simplifies to give the dispersion relation (2). (In (11), we assumed  $k > 0$ ; modifying (1) to  $\omega^2 = g|k|$  removes this restriction).

Form (11) (and recalling that  $\frac{\partial \phi}{\partial y} = v$ ,  $\frac{\partial \phi}{\partial x} = u$ ), we deduce that individual fluid particles follow circular paths. The size of these paths decrease exponentially with depth. Already at a

depth of only one wavelength  $\lambda = \frac{2\pi}{k}$ , they are reduced by a factor of  $e^{-2\pi} \approx 0.002$ . In this limit of small wave amplitudes, there is no net fluid transport.

### 4.3.2 Group speed vs. phase speed

In deep water, low amplitude waves

$$\phi(x,t) = a \cos(kx - \omega t)$$

with a single frequency  $k$  have  $\omega = \sqrt{gk}$  and travel with the speed  $c_p = \sqrt{g/k}$ . Figure 2 illustrates such a wave train and also its representation in Fourier space - just the single frequency  $k$  present.

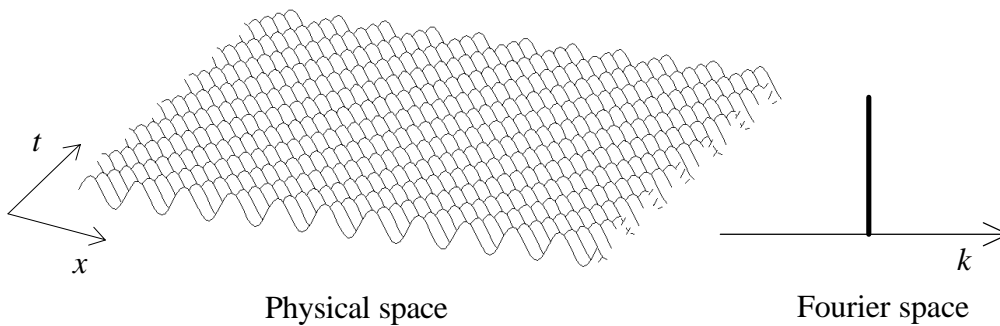


Figure 2. Physical and Fourier space representations of a uniform wave train.

In many situations, waves travel in *wave packets*, as sketched in Figure 3. In Fourier space, a wave packet is a superposition of waves with very similar frequencies.

Already the sum of just two close frequencies and no time dependence leads to a string of repeating wave packets;  $\sin(k+\Delta k)x + \sin(k-\Delta k)x = 2 \sin kx \cos \Delta k x$ . In the RHS,  $\sin kx$  is a fast oscillation and  $\cos \Delta k x$  gives a slow modulation of the amplitude. With more tightly clustered wave modes, the result can be a single wave packet (rather than a string of them).

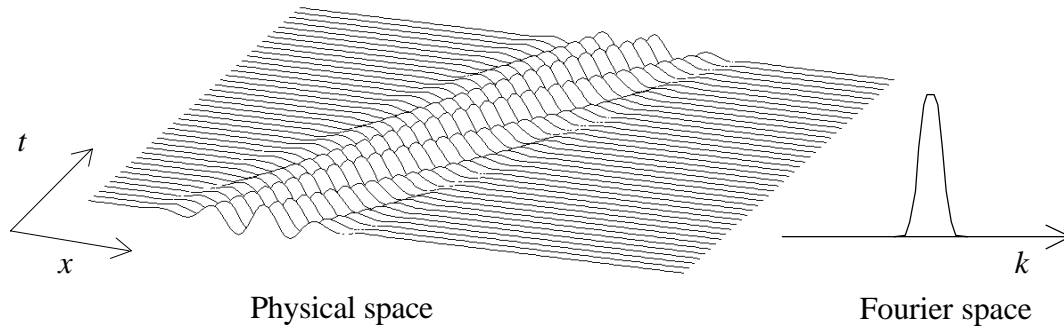


Figure 3. Physical and Fourier space representations of a wave packet (Fourier picture shows schematically the *envelope* of  $|\hat{\phi}(k)|$ ).

In the progressing wave packet in Figure 3, we can clearly note two entirely different frequencies:

- the whole group moves with a *group speed*  $c_g$ , and
- individual crests move with the *phase speed*  $c_p$ .

The relation between these two different speeds can be derived from the dispersion relation:

$$\text{Group speed:} \quad c_p = \frac{\omega(k)}{k} \quad (1)$$

$$\text{Phase speed:} \quad c_g = \frac{d\omega(k)}{dk} \quad (2)$$

The formula for  $c_g$  can be obtained as follows: The wave function for a single wave number  $k_0$  can be written

$$\phi(x, t) = \hat{\phi}_0 e^{i(k_0 x - \omega(k_0) t)} \quad (3)$$

A wave packet with the same main wave number is similarly a superposition of different waves

$$\phi(x, t) = \int_{-\infty}^{\infty} \hat{\phi}(k) e^{i(kx - \omega(k) t)} dk$$

where  $\hat{\phi}(k)$  is very near zero everywhere but has a sharp peak at  $k = k_0$ . In that small neighborhood of  $k_0$ , we have

$$\omega(k) \approx \omega(k_0) + (k - k_0) \cdot \alpha \quad \text{where} \quad \alpha = \left. \frac{d\omega}{dk} \right|_{k=k_0}.$$

Hence

$$\phi(x, t) \approx \int_{-\infty}^{\infty} \hat{\phi}(k) e^{i(kx - (\omega(k_0) + (k-k_0)\alpha)t)} dk =$$

$$= \underbrace{e^{i(k_0 x - \omega(k_0)t)}}_{\text{Pure harmonic of wave number } k_0 \text{ (just as in (3))}} \cdot \underbrace{\int_{-\infty}^{\infty} \hat{\phi}(k) e^{i(k-k_0)(x-\alpha t)} dk}_{\text{Modulation factor providing envelope form; } x \text{ and } t \text{ appear here only in the combination } x - \alpha t \text{ showing that this expression translates with speed } \alpha (= d\omega/dk|_{k=k_0}); \text{ hence this is the group speed.}}$$

Pure harmonic  
of wave number  
 $k_0$  (just as in (3))

Modulation factor providing envelope form;  $x$  and  $t$  appear here only in the combination  $x - \alpha t$  showing that this expression translates with speed  $\alpha$  ( $= d\omega/dk|_{k=k_0}$ ); hence this is the group speed.

Table 1 summarizes the dispersion relations and the two speeds for some different types of waves (we assume the liquid to be water with density  $\rho = 1$ ).

TABLE 1. Dispersion relation and wave speed for some different wave types

Type of wave	Dispersion relation $\omega =$	$c_p = \frac{\omega}{k}$	$c_g = \frac{d\omega}{dk}$	$c_g/c_p$	Comment
Gravity wave, deep water	$\sqrt{gk}$	$\sqrt{\frac{g}{k}}$	$\frac{1}{2}\sqrt{\frac{g}{k}}$	$\frac{1}{2}$	$g =$ acceleration of gravity
Gravity wave, shallow water	$\sqrt{gk \tanh kh}$	$\sqrt{\frac{g}{k} \tanh kh}$	product $c_p \cdot (c_g/c_p)$	$\frac{1}{2} + \frac{hk}{\sinh(2hk)}$	$h =$ water depth
Capillary wave	$\sqrt{Tk^3}$	$\sqrt{Tk}$	$\frac{3}{2}\sqrt{Tk}$	$\frac{3}{2}$	$T =$ surface tension
Particle wave (in quantum mechanics; free particle in 1-D)	$\frac{hk^2}{4\pi m}$	$\frac{hk}{4\pi m}$	$\frac{hk}{2\pi m}$	2	$h =$ Planck's constant, $m =$ particle mass
Light in vacuum	$ck$	$c$	$c$	1	$c = 299,792,458$ m/s
Light in a transparent medium	$\frac{ck}{n(k)}$	$\frac{c}{n(k)}$	$c_p \left(1 - \frac{kn'(k)}{n(k)}\right)$	$1 - \frac{kn'(k)}{n(k)}$	$n(k)$ index of refraction of medium

The results in this table have some notable implications:

- Since  $c_p > c_g$  for gravity waves, a surfer gets the longest ride if he/she can catch a wave at the end of a group,
- If a straw or twig sticks up in a stream, the phase gets locked in at it. Small wavelength capillary waves extend upstream (since for them  $c_g > c_p$ ) whereas longer wavelength gravity waves extend in a sector downstream,
- The ratio  $c_g/c_p = 1/2$  for gravity waves can be shown to imply that the distinct V-shaped wake left behind it always forms an angle  $19.5^\circ$  ( $\arcsin 1/3$ ) with the center line *independently of the speed of the ship*,

- In quantum mechanics, a particle's position is undetermined within the width  $\Delta x$  of its wave packet. Fourier analysis will show that  $\Delta x$  and its spread in Fourier space  $\Delta k$  are related by  $\Delta x \cdot \Delta k > 1$  (or  $\Delta x \cdot \Delta k > \text{constant}$ ; there is some arbitrariness in how wide one regards a Gaussian pulse to be). De Broglie's relation  $k = \frac{2\pi mv}{h}$  relates wavenumber  $k$  to momentum  $p = mv$  and Planck's constant  $h$ ; here  $m$  is the particle mass and  $v$  ( $= c_g$ ) its velocity. From this, we obtain Heisenberg's uncertainty relation  $\Delta x \cdot \Delta v > \frac{h}{2\pi m}$ . In other words, the product of the uncertainties in a particle's position and velocity must always exceed  $\frac{h}{2\pi m}$ .

### 4.3.3 Stokes' waves:

These are spatially periodic, translating solutions  $\eta(x, t) = \eta(x - ct)$  for the surface elevation (again in the case of infinite depth and no surface tension; gravity is assumed to be the only restoring force). However, we no longer consider only infinitesimal amplitudes, so the approximation in equation (1)  $\eta(x, t) = a \cos(kx - \omega t)$  amounts only to the first term in an expansion for small  $a$ . Including terms up to  $a^3$  can be shown to give

$$\eta(x, t) \approx a \cos(kx - \omega t) + \frac{1}{2}ka^2 \cos 2(kx - \omega t) + \frac{3}{8}k^2a^3 \cos 3(kx - \omega t) \quad (12)$$

where

$$\omega \approx \sqrt{gk} \left( 1 + \frac{1}{2}k^2a^2 \right)$$

is a next-order approximation beyond (2) for the dispersion relation.

Some observations:

- Since the governing equations (6)-(8) are nonlinear, solutions cannot be superposed (or multiplied by scalars) to give other ones.
- In real water, high Stokes waves are never seen. Such a wave is (near the top) unstable to highly oscillatory infinitesimal disturbances. Another (physically even more important) instability arises already at low amplitudes - the *Benjamin-Feir instability* causes uniform periodic wave trains to lose their periodicity. The wave heights will always be irregular (and time dependent).
- A Taylor expansion will fail to converge at a radius determined by the nearest singularity - this may well be an unphysical one (e.g. at a negative or complex point for a variable that physically must always be positive). The expansion (12) can be shown to diverge - if continued to more terms - before the highest Stokes' wave is reached.
- In the limit of large amplitude, already Stokes (1880) showed that the wave would have a peaked top of  $120^\circ$  angle. Expansions valid near this limit are very complicated (many wave properties feature an infinity of oscillations as the maximum height wave is approached).
- Including terms of higher order than  $a^3$  in (12) will not only increase the number of Fourier modes; the coefficients for the individual modes will turn into power series expansions in  $a$ .