4.4 First Order Systems for some Linear Wave Equations

In this Section, we first see how several linear wave equations can be written as a system of first order PDEs. Such formulations are very useful both for analysis and numerical solution. As an example of the former, we determine what steadily translating wave solutions some of the equations admit.

4.4.1 Formulation of some Linear Wave Equations as First Order Systems

<u>1-D acoustic wave equation:</u>

The 1-D acoustic wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (obtained for two different types of waves on a string in Section ii.4.2) can be written in a few different ways, e.g.

i. $\left(\frac{\partial}{\partial t} + c\,\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\,\frac{\partial}{\partial x}\right)u = 0$, and

ii. $\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} = c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} u \\ v \end{bmatrix}$ Here, an extra variable v is introduced, satisfying $v_t = c\,u_x$. (1)

Although version i. gives some insights:

> The general solution to the two factors being zero independently are $u(x,t)=F(x-ct)$ and $u(x,t)=G(x+ct)$ resp. with F and G arbitrary functions of one scalar variable. The solution to the original equation must allow both of these possibilities; its general solution is indeed $u(x,t) = F(x-ct)+G(x+ct)$,

such a factorization is rarely applicable. On the other hand, the second version is both very useful and generally available.

<u>2-D acoustic wave equation:</u>

For a homogeneous medium, the equation is again the same for transverse- and longitudinal motions:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

To write this as a first order system, we need to introduce 2 auxiliary variables *v* and *w*, giving

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = c\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} u \\ v \\ w \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\frac{\partial}{\partial y}\begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad (2)$$

<u>2-D elastic wave equation:</u>

If a 2-D plate also possesses elastic properties, the governing equations (for motions within the *x*,*y*- plane) takes the form

$$\left\{\begin{array}{rcl} \frac{\partial u}{\partial t} & = & \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \\ \frac{\partial v}{\partial t} & = & \frac{\partial g}{\partial x} + \frac{\partial h}{\partial y} \\ \frac{\partial f}{\partial t} & = & (\lambda + 2\mu)\frac{\partial u}{\partial x} + \lambda\frac{\partial v}{\partial y} \\ \frac{\partial g}{\partial t} & = & \mu\frac{\partial v}{\partial x} + \mu\frac{\partial u}{\partial y} \\ \frac{\partial h}{\partial t} & = & \lambda\frac{\partial u}{\partial x} + (\lambda + 2\mu)\frac{\partial v}{\partial y} \end{array}\right. .$$

Here,	u,v	displacements in *x*- and *y*-directions of a marker element,
	f,g,h	*x*-compression, shear, and *y*-compression resp. of marker element,
	λ,μ	elastic constants (with respect to compression and shear).

This can immediately be stated in matrix form:

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \\ f \\ g \\ h \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \lambda+2\mu & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} u \\ v \\ f \\ g \\ h \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \lambda & 0 & 0 & 0 \\ \mu & 0 & 0 & 0 & 0 \\ 0 & \lambda+2\mu & 0 & 0 & 0 \end{bmatrix}\frac{\partial}{\partial y}\begin{bmatrix} u \\ v \\ f \\ g \\ h \end{bmatrix}. \qquad (3)$$

<u>3-D Maxwell's equations:</u>

For a loss-less medium, these equations are usually written as

$$\left\{\begin{array}{rcl} \frac{\partial E_x}{\partial t} & = & \frac{1}{\varepsilon}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) \\ \frac{\partial E_y}{\partial t} & = & \frac{1}{\varepsilon}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) \\ \frac{\partial E_z}{\partial t} & = & \frac{1}{\varepsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \end{array}\right. , \qquad \left\{\begin{array}{rcl} \frac{\partial H_x}{\partial t} & = & \frac{1}{\mu}\left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}\right) \\ \frac{\partial H_y}{\partial t} & = & \frac{1}{\mu}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) \\ \frac{\partial H_z}{\partial t} & = & \frac{1}{\mu}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right) \end{array}\right. \qquad (4)$$

where	E_x, E_y, E_z	the components of the electric field,
	H_x, H_y, H_z	the components of the magnetic field,
	μ	permeability,
	ε	permettivity.

(For lossy media, we need to subtract σE_x, σE_y, σE_z, ρH_x, ρH_y, ρH_z resp. from the six RHSs; here σ denotes conductivity and ρ magnetic resistivity).

Like in the 2-D elastic case, the matrix form follows immediately from the first order equations.

4.4.2 Determination of wave speeds

1-D acoustic wave equation:

We look for translating solutions to (1) of the form $\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} = \begin{bmatrix} u(t-\alpha x) \\ v(t-\alpha x) \end{bmatrix}$, translating with a speed of $1/\alpha$. Then $\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}'$ and $\frac{\partial}{\partial x}\begin{bmatrix} u \\ v \end{bmatrix} = -\alpha \begin{bmatrix} u \\ v \end{bmatrix}'$. Substituting into (1) gives

$$\begin{bmatrix} u \\ v \end{bmatrix}' = -c\alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}',$$

i.e. such solutions exist if $\begin{bmatrix} 1 & c\alpha \\ c\alpha & 1 \end{bmatrix}$ is singular. From

$$0 = \det \begin{bmatrix} 1 & c\alpha \\ c\alpha & 1 \end{bmatrix} = 1 - c^2 \alpha^2$$

follows $\alpha_{1,2} = \pm 1/c$. We conclude that (1) admits translating solutions with speeds $1/\alpha_{1,2} = \pm c$. This is in complete agreement with our (much stronger) statement above that the general form of the solution to the 1-D wave equation can be written as $u(x,t) = F(x-t) + G(x+t)$.

2-D Acoustic Wave Equation:

We confine ourselves again to look for translating solutions, now of the form $\begin{bmatrix} u(x,y,t) \\ v(x,y,t) \\ w(x,y,t) \end{bmatrix} = \begin{bmatrix} u(t-\alpha x-\beta y) \\ v(t-\alpha x-\beta y) \\ w(t-\alpha x-\beta y) \end{bmatrix}$. This solution is constant along lines in the $(-\beta, \alpha)-$direction, and moves in the $(\alpha,\beta)-$direction with the velocity $1/\sqrt{\alpha^2+\beta^2}$. From $\frac{\partial}{\partial t}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}u\\v\\w\end{bmatrix}'$, $\frac{\partial}{\partial x}\begin{bmatrix}u\\v\\w\end{bmatrix}=-c\alpha\begin{bmatrix}u\\v\\w\end{bmatrix}'$, $\frac{\partial}{\partial y}\begin{bmatrix}u\\v\\w\end{bmatrix}=-c\beta\begin{bmatrix}u\\v\\w\end{bmatrix}'$ follows now

$$\begin{bmatrix}u\\v\\w\end{bmatrix}=-c\,\alpha\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}'-c\,\beta\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}'$$

with non-trivial solutions only if

$$0=\det\begin{bmatrix}1&c\alpha&c\beta\\c\alpha&1&0\\c\beta&0&1\end{bmatrix}=1-c^2(\alpha^2+\beta^2)\quad .$$

This shows that there is no 'preferred direction' in the (x,y)-plane; there are solutions which translate with speed c in any direction.

2-D Elastic Wave Equation:

Analogously to the previous case, again looking for translating solutions leads us now to consider

$$0=\det\begin{bmatrix}1&0&\alpha&\beta&0\\0&1&0&\alpha&\beta\\(\lambda+2\mu)\alpha&\lambda\beta&1&0&0\\\mu\beta&\mu\alpha&0&1&0\\\lambda\alpha&(\lambda+2\mu)\beta&0&0&1\end{bmatrix}=$$

$$=\lfloor\mu(\alpha^2+\beta^2)-1\rfloor\cdot\lfloor(\lambda+2\mu)(\alpha^2+\beta^2)-1\rfloor.$$

There are now two types of possible waves, both with velocities that are direction independent:

P (pressure) - wave: $v_p=(\lambda+2\mu)^{1/2}$, and
S (shear) - wave: $v_s=\mu^{1/2}$.