4.5 Analytical Solutions of the Acoustic Wave Equations

We have already come across the acoustic wave equation in one and two dimensions. In n dimensions, it takes the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} \right) \qquad (1)$$

This is sometimes more compactly written as $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$ where $u(\mathbf{x},t)$ is a scalar function of $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ and $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$. We let ∇^2 stand for $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$.

Since (1) has a second derivative in time, we will need two initial conditions to solve it forward in time:

$$\begin{cases} u(\mathbf{x},0) &= f(\mathbf{x}) \\ u_t(\mathbf{x},0) &= g(\mathbf{x}) \end{cases} \quad . \qquad (2)$$

In the subsections below, we first discuss the solution to (1) in one dimensions, and then make some general observations about the solution in higher dimensions.

4.5.1 Acoustic Wave Equation in 1-D; d'Alembert's Solution

For both transverse and longitudinal waves, we found the constant medium 1-D wave equation to take the form

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad . \qquad (3)$$

The general solution on an infinite interval can be written as

$$u(x,t) = F(x - ct) + G(x + ct) \qquad (4)$$

where F and G are arbitrary functions. Verification of this is straightforward:

$$u_{xx} = F''(x - ct) + G''(x - ct) \quad , \text{ and}$$
$$u_{tt} = c^2 F''(x - ct) + c^2 G''(x - ct) \quad ;$$

hence (3) is satisfied. To solve (3) forward in time, we also need two initial conditions. In 1-D, we write (2) as

$$\begin{cases} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x) \end{cases} \quad . \tag{5}$$

D'Alembert's solution for this initial value problem is

$$u(x,t) = \frac{1}{2}\Big[f(x-ct) + f(x+ct) + \frac{1}{c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi \Big] \tag{6}$$

Derivation 1:

The idea is to convert $f(x)$ and $g(x)$ from (5) into the functions $F(x)$ and $G(x)$ that appeared in (4). From (4) and (5)

$$\begin{cases} f(x) &= u(x,0) &= F(x) + G(x) & \text{(7 a)} \\ g(x) &= u_t(x,0) &= -c F'(x) + c G'(x) & \text{(7 b)} \end{cases} \quad .$$

Differentiating (7 a) and multiplying it by c gives

$$c f'(x) = c F'(x) + c G'(x) \; . \tag{8}$$

Adding and subtracting (7 b) and (8) give

$$\begin{cases} c F'(x) &= \frac{1}{2}[c f'(x) - g(x)] \\ c G'(x) &= \frac{1}{2}[c f'(x) + g(x)] \end{cases}$$

which we integrate:

$$\begin{cases} c F(x) &= \frac{1}{2}\Big[c f(x) - \int_0^x g(\xi)\, d\xi \Big] + c_1 \\ c G(x) &= \frac{1}{2}\Big[c f(x) + \int_0^x g(\xi)\, d\xi \Big] + c_2 \end{cases} \quad .$$

Hence

$$u(x,t) = F(x-ct) + G(x+ct) =$$

$$= \frac{1}{2}\Big[f(x-ct) - \frac{1}{c}\int_0^{x-ct} g(\xi)\, d\xi \Big] + \frac{1}{2}\Big[f(x+ct) + \frac{1}{c}\int_0^{x+ct} g(\xi)\, d\xi \Big] + c_3 =$$

$$= \frac{1}{2}\Big[f(x-ct) + f(x+ct) + \frac{1}{c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi \Big] + c_3 \, . \tag{9}$$

Finally, considering $t = 0$, we get $c_3 = 0$, and we have therefore expressed $u(x,t)$ in terms of the initial conditions $f(x)$ and $g(x)$.

Derivation 2:

Since (3) has constant coefficients and the space interval is [-∞,∞], it is natural to try to express $u(x,t)$ in its Fourier transform

$$u(x,t) = \int_{-\infty}^{\infty} \hat{u}(\omega,t)\, e^{i\omega x}\, d\omega \quad . \tag{10}$$

Substituting this into (3) gives $\int_{-\infty}^{\infty} (\frac{\partial^2 \hat{u}}{\partial t^2} + c^2 \omega^2 \hat{u})\, e^{i\omega x} d\omega = 0$ for all x, implying

$$\frac{\partial^2 \hat{u}}{\partial t^2} + c^2 \omega^2 \hat{u} = 0 \qquad .$$

The key idea with the Fouriet transform (10) is that we now have an ODE instead of a PDE. In the t-direction, it has constant coefficients. So

$$\hat{u}(\omega, t) = d_1(\omega)\, e^{i c \omega t} + d_2(\omega)\, e^{-i c \omega t}$$

where $d_1(\omega)$ and $d_2(\omega)$ are constants with respect to t (but may well depend on ω). The general solution to (3) is therefore

$$u(x,t) = \int_{-\infty}^{\infty} \left[d_1(\omega)\, e^{i\omega(x+ct)} + d_2(\omega)\, e^{i\omega(x-c\,t)} \right] d\omega \quad . \tag{11}$$

It now remains to find $d_1(\omega)$ and $d_2(\omega)$ so the equations in (5) are satisfied, i.e.

$$f(x) = u(x,0) = \int_{-\infty}^{\infty} [d_1(\omega) + d_2(\omega)]\, e^{i\omega x}\, d\omega \tag{12}$$

and

$$g(x) = \left.\frac{\partial u}{\partial t}\right|_{t=0} = \int_{-\infty}^{\infty} [i\,\omega\, c\, d_1(\omega) - i\,\omega\, c\, d_2(\omega)]\, e^{i\omega x}\, d\omega \qquad \text{, i.e.}$$

$$h(x) = \int_{-\infty}^{x} g(\xi) d\xi = \int_{-\infty}^{\infty} [\, c\, d_1(\omega) - c\, d_2(\omega)]\, e^{i\omega x}\, d\omega \qquad . \tag{13}$$

By linear combinations of (12) and (13) follows

$$\frac{1}{2}\left[f(x) + \frac{1}{c} h(x) \right] = \int_{-\infty}^{\infty} d_1(\omega)\, e^{i\omega x}\, dx \qquad ,$$
$$\frac{1}{2}\left[f(x) - \frac{1}{c} h(x) \right] = \int_{-\infty}^{\infty} d_2(\omega)\, e^{i\omega x}\, dx \qquad .$$

By substituting $x \to x + c\,t$ and $x \to x - c\,t$ in the two equations above, we can recover the RHS of (11):

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + \frac{1}{c} h(x+ct) \right] + \frac{1}{2}\left[f(x-ct) - \frac{1}{c} h(x-ct) \right] =$$

$$= \frac{1}{2}\left[f(x+ct) + f(x-ct) + \frac{1}{c} \int_{x-ct}^{x+ct} g(\xi)\, d\xi \right] \qquad \square$$

Example: Determine the solution to the initial-boundary value problem

PDE: $u_{tt} - c\, u_{xx} = 0$,

IC: $\begin{cases} u(x,0) = f(x) & \text{positive pulse with compact support,} \\ u_x(x,0) = 0 \end{cases}$

BC: $\begin{cases} \text{case a: } u(0,t) = 0 & \text{or} \\ \text{case b: } u_x(0,t) = 0 \ . \end{cases}$

Solution: For both cases, we extend the domain to $[-\infty,\infty]$ by defining for $x < 0$:

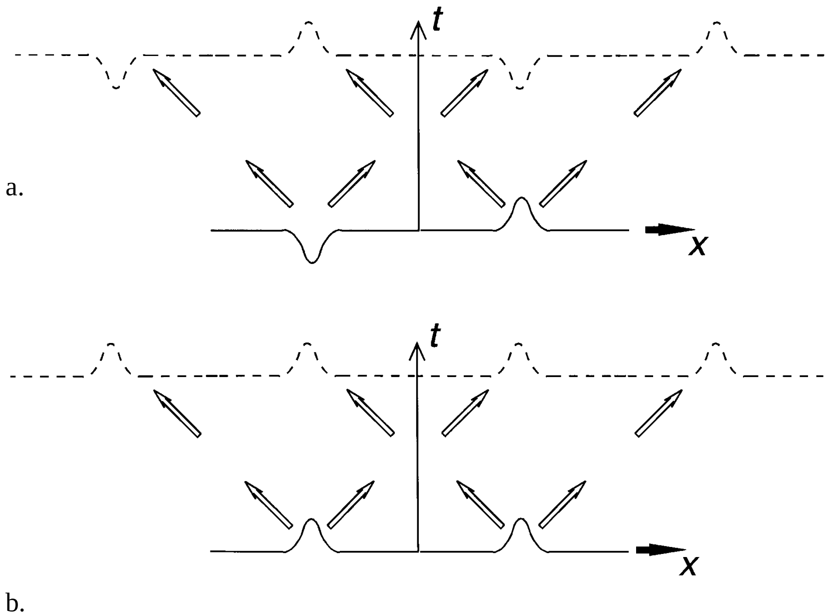
a. $u(x,0) = f(-x)$, and b. $u(x,0) = - f(-x)$.

For the pure initial value problems we now get, the solution becomes (from (6), with $g(x) = 0$)

$$u(x,t) = \frac{1}{2}[u(x-ct,0)+u(x+ct,0)] \quad ,$$

i.e. each initial pulse splits in two parts, traveling in opposite directions. By symmetry, the BCs are seen to hold at $x = 0$. We get the solutions to the original problem by inspecting the solutions to the extended one for $x \geq 0$, $t \geq 0$. As Figure 1 a,b show, the effect of the BCs are to reflect any incoming wave, in case a swapping its sign, and in case b preserving it.

A wall that somehow averages the effect of the two boundary conditions will become a perfect sound absorber.



Figures 1 a,b. Schematic solutions to parts a and b obtained for $x \geq 0$, $t \geq 0$.

□

4.5.2 Acoustic wave equation in higher dimensions

A closed form general solution to (1) with initial conditions (2), reminiscent of (6), can again be written down in higher diomensions (but we will not do it here). Several key properties of the solutions to (1), (2) turn out to differ significantly dependent on the number of dimensions:

1-D The general solution, given by (6), supports two very different kinds of wave motions:

i. Cleanly translating pulses (if $f\,(x) \neq 0$, $g(x) = 0$), leaving a zone of silence behind, and

ii. disturbances which spread out across a complete interval (if $f\,(x) = 0$, $g(x) \neq 0$),

2-D (and higher *even* dimensions) Both solution types (i.e. initializing (1) with $f\,(\boldsymbol{x}) \neq 0$, $g(\boldsymbol{x}) = 0$ and $f\,(\boldsymbol{x}) = 0$, $g(\boldsymbol{x}) \neq 0$ respectively) fall in the second category, i.e. they never leave any zone of silence behind (as can be seen when dropping small pebbles in a pond),

3-D (and higher *odd* dimensions) The situation is completely reversed from 2-D - both solution types now leave a zone of perfect silence behind any traveling wave front. This remarkable property makes speech possible in 3-D. An outgoing sound signal translates away from the source, and attenuates with the distance traveled, but undergoes no other change (in 5-D, 7-D etc., this last property is not quite true - more about that later).

Figure 2 illustrates the important concepts of *domain of dependence* and *region of influence* in these cases.

These equations all feature well defined, finite singal speeds, as we saw for 1-D and 2-D in Section ii.4.4.2, where we considered planar wave fronts. However, one is often more interested in waves that expand with equal strength in all directions away from a point source. To analyze that situation, we start by noting that a solution $u(x_1,x_2,\ldots,x_n,t)$ to the n-D wave equation (1)

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} \tag{14}$$

which depends only on $r = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$ and t will satisfy

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r} \quad . \tag{15}$$

<u>Derivation of (10)</u>:

The chain rule gives

$\frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x_i} = \frac{\partial u}{\partial r}\cdot x_i \cdot \frac{1}{r}$ (noting that $r^2 = x_1^2 + x_2^2 + \ldots + x_n^2$ implies $2\,r\,\frac{\partial r}{\partial x_i} = 2\,x_i$), and
$\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right) = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cdot\frac{x_i}{r}\right)\cdot\frac{x_i}{r} = \frac{\partial^2 u}{\partial r^2}\left(\frac{x_i}{r}\right)^2 + \frac{\partial u}{\partial r}\cdot\frac{1}{r} - \frac{\partial u}{\partial r}\cdot\frac{x_i^2}{r^3}$. Summation over i now gives
$\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r}$. □

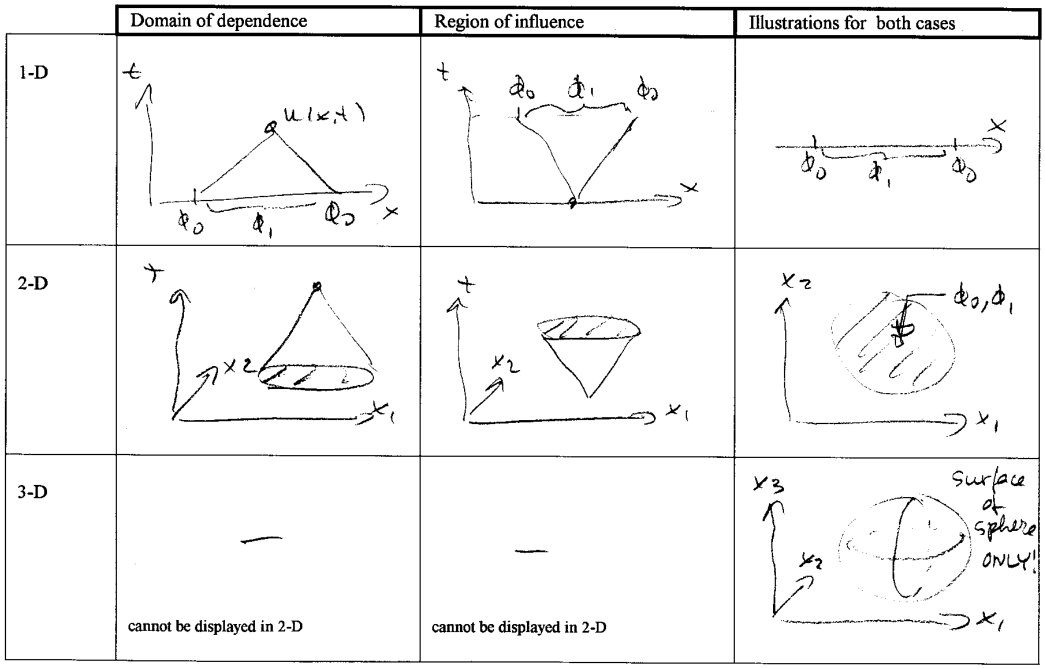


Figure 2. Illustration of *domains of dependence* and *regions of influence* for solutions to the wave equation in 1, 2, and 3 D.

To illustrate some of the points we just made about solutions in different numbers of dimensions, we determine next for which values of n equation (10) posesses solutions which simply translate and decay, but undergo no other changes. Such solutions must be of the form

$$u(r,t) = \alpha(r) \cdot f(t - \delta(r)) \qquad (11)$$

where

$\alpha(r)$	corresponds to attenuation (satisfying $\alpha(r) > 0;\ \alpha(1) = 1$),
$\delta(r)$	represents a time delay ($\delta(r) > 0;\ \delta(0) = 0$), and
$f(t)$	is any arbitrary function (representing the sound/voice signal to be transmitted).

The analysis below shows that solutions of the form (11) exist only in 1-D and 3-D. This is consistent with our observation that 3-D is indeed special in terms of supporting perfect sound propagation.

Verification that solutions of the form (11) only are possible in 1-D and 3-D:

Differentiation of (11) gives

$$\frac{\partial^2 u}{\partial t^2} = \alpha f'' \ , \ \frac{\partial u}{\partial r} = \alpha' f - \alpha \delta' f' \quad \text{and} \quad \frac{\partial^2 u}{\partial r^2} = \alpha'' f - \alpha' \delta' f' - (\alpha' \delta' + \alpha \delta'') f' + \alpha \delta'^{\,2} f''.$$

We substitute these values into (10):

$$\frac{\alpha}{c^2} f'' = \alpha'' f - \alpha' \delta' f' - (\alpha' \delta' + \alpha \delta'') f' + \alpha \delta'^{\,2} f'' + \frac{n-1}{r}(\alpha' f - \alpha \delta' f').$$

For this is to hold for all (differentiable) functions f, the coefficients for each of f, f' and f'' must vanish. From the coefficient for f'' follows $\alpha \delta'^{\,2} = \frac{\alpha}{c^2}$, i.e. $\delta = \frac{r}{c}$, $\delta' = \frac{1}{c}$ and $\delta'' = 0$. With use of this, the coefficients for f and f'' give

$$\alpha'' + \frac{n-1}{r}\alpha' = 0 \qquad \text{and} \qquad \frac{1}{c}\left(2\alpha' + \frac{n-1}{r}\alpha\right) = 0$$

with solutions of the form $\alpha = K r^{\beta}$ where K and β are constants. The two equations now become

$$\beta(\beta - 1) + (n - 1)\beta = 0 \qquad \text{and} \qquad 2\beta + (n - 1) = 0 \ \text{ resp.},$$

with solutions for β only if $n = 1$ and $n = 3$. In the former case, $\alpha(r) = 1$ and there is no attenuation. In 3-D, $\beta = -1$, and sound strength decays like $\alpha(r) = 1/r$.

□

A natural question to ask is what happens in 5-D, 7-D etc. Although (10) for all *odd* dimensions allow solutions that are 'compact pulses' - exaxtly zero outside any arbitrarily small interval - for $n \geq 5$, such pulses do not simply decay in amplitude with the traveled distance (as is required by (11)), but they undergo some other minor changes as well. For example, the 1-D solution to (10)

$$u(r,t) = f(t - \tfrac{r}{c})$$

corresponds in 3-D to

$$u(r,t) = \tfrac{1}{r} f(t - \tfrac{r}{c}) \qquad .$$

In 5-D, the form assumed in (11) is no longer the right one. It is easily verified that (10), for arbitrarty f , is now satisfied by

$$u(r,t) = \tfrac{1}{r^3} f(t - \tfrac{r}{c}) - \tfrac{1}{c r^2} f'(t - \tfrac{r}{c}) \qquad .$$

A traveling compact pulse again stays compact, but its time evolution is no longer just a translation and an attenuation.