Chapter 1

LINEAR SYSTEMS, QR AND SVD

A general linear system $A\,x = b$ of m equations in n unknowns can graphically be displayed as

$$\begin{bmatrix} \\ A \\ \\ \end{bmatrix}_{m\times n}\begin{bmatrix} \\ x \\ \\ \end{bmatrix}_{n} = \begin{bmatrix} \\ b \\ \\ \end{bmatrix}_{m} .$$

A key issue is to determine whether such a system has 0, 1 or ∞ many solutions (these turn out to be the only possibilities). Section 1.1 presents a (primarily theoretical) procedure to determine the solution set - reducing the matrix to *echelon form*. The most important case is probably when there is one solution - *Gaussian elimination* is then appropriate (Section 1.2). The case with no solution is also ubiquitous - it arises for ex. if more measurements are made in an experiment than there are free parameters to be determined. The *least-squares problem* then amounts to finding the vector x that minimizes the error $\|Ax - b\|$.

Both for solving linear systems, as well as for other applications, it is often important to decompose a matrix A as $A = QR$ - the product of a *unitary* matrix Q and an *upper triangular* matrix R. *QR-decompositions* are described in Section 1.3, followed by a discussion of the least-squares problem (for over-determined systems) in Sections 1.4. In the final Section 1.5 of this chapter, we consider another unitary matrix-based way to write a matrix A which also has many applications: $A = Q_1 D Q_2$ where $Q_{1,2}$ are unitary and D is *diagonal*. Applications of this *singular value decomposition* (SVD)-form have increased rapidly in the last decade.