Chapter 1

INTRODUCTION

The subject of Applied Mathematics includes a large collection of analytical techniques, mostly developed in the 19th century (such as complex variables, transforms, basic theory for ODEs and PDEs) together with numerous more modern refinements (e.g. matched asymptotic expansions). More recently, it has also come to include an extensive body of numerical methods. Although the analytical tools of applied mathematics are essential both to obtain concise formulations of most problems, and to gain many insights into their solutions, it is only in rare cases that these tools alone suffice in providing all the information (visualizations etc.) that are needed.

Numerical analysis concerns the creation of computational algorithms, their analysis, and their usage. Numerical methods are unparalleled in power and flexibility. For example, they can provide close approximations to any well-posed system of PDEs in ways which are well suited for visualization etc., a far cry from the scope of exact methods.

The 'life-blood' of all parts of applied mathematics is applications. In spite of that, both analytical and numerical techniques are often taught in very 'encyclopedic' fashions (and as two entirely separate subjects). Key techniques are often enumerated in manners, which are better suited for reference works than for first introductions. Instead, this book will give examples of how one carries through all the steps: problem →mathematical formulation →theoretical analysis →numerical solution.

A good familiarity with calculus is assumed, and some knowledge of basic analytical and numerical techniques is helpful. It is also assumed that the reader has completed a first course in ordinary differential equations. Current ODE courses often include a number of modeling exercises that lead to ODEs which then always can be solved numerically, and sometimes also analytically. This book does not provide further examples of this kind but extends the modeling idea to situations where the process involves a wider mix of steps and techniques. Although the present book is mainly written for readers nearing completion of their undergraduate studies, the material can also be studied while taking first courses in Methods of Applied Mathematics and in Numerical Analysis.

The general plan for this book is as follows:

1. We discuss techniques only according to what we need to have available in order to resolve issues that arise. The primary modeling problems we consider - all currently active research areas - have been selected in such a way that, between them, they involve quite a broad range of methods:

   (a) Tomographic image reconstruction (commonly used for ex. in medical imaging),
   (b) Freak ocean waves - isolated giant waves that can severely damage (and even sink) large ships, such as supertankers. Although partly random in nature, forecasting may become a possibility,
   (c) Automated facial recognition,
   (d) Global Positioning System (GPS) satellite navigation - a revolutionary system for land-, sea- and air navigation,
   (e) Seismic forward modeling - a key procedure in exploring for oil and gas.

2. We will not always try to follow some fine middle path between the opposite sins of mathematical nitpicking and loose handwaving - the latter path is far more typical of how one in practice 'goes about' exploring problems that arise. Common-sense practical approaches and heuristic arguments are preferred over rigorous technical proofs (unless the latter adds key insights in how, why, and when something works).

3. Especially for a one-semester course, there should be no rush to cover all the material that is included in these notes.

This book has two main purposes:

1. To illustrate key phases of mathematical modeling:
   (a) Bring the problem to a form in which the tools of applied mathematics can be brought to bear. A number of simplifying assumptions are normally needed to arrive at governing equations,
   (b) Carry out analysis, as far as is practical. This will increase the understanding of the phenomenon, but virtually always falls far short of any closed-form solutions or illuminating graphycal displays. Hence:
   (c) Pursue either or (preferrably) both of the approaches below:
      i. Make analytic simplifications on the governing equations and, by finding closed-form solutions or - more likely - by perturbation analysis or asymptotic techniques, gain insights into special cases, and
      ii. Apply numerical techniques. These will likely to give accurate results in almost all cases, typically in the form of graphical displays and/or statistical information.

2. To provide a motivational, non-rigorous introduction to a large number of key ideas and techniques in the field of applied mathematics.

Modeling is normally a highly iterative procedure; the insights gained by analysis and numerics is used to improve the earlier steps, such as the design of governing equations. Or the modeling results may feed back into changes in the very process that is studied. In industry, manufacturing techniques, product re-designs, and customer feedbacks are all likely to be part of the interactive modeling cycle.

It is inevitable that any modeling book, written around only a handful of applications, will have to be very incomplete in many ways. Although the applications were selected to touch on a relatively broad area of both analytical- and numerical techniques, several important ones are not even mentioned. In the area of analysis, this includes many topics on differential equations (such as eigenmode expansions, stability of solutions, and chaos). Among numerical methods, there is for example no mention of finite elements. The subject of statistics is only briefly touched on.