Lecture for October 25, 2004

Birth-and-Death Processes

Let $\{X(t)\}_{t\geq 0}$ be a Markov chain. Then $\{X(t)\}$ is called a birth-and-death process if,as $h \searrow 0$,

$$P(X(t+h)-X(t)=k|X(t)=i)=\begin{cases}\lambda_i h+o(h), & \text{if } k=1\\ \mu_i h+o(h), & \text{if } k=-1\\ o(h) & \text{if } |k|>1\end{cases}$$

Things to note:

1. The last condition,

   $$\text{“}o(h) \;\text{ if }\; |k|>1\text{“},$$

   implies that pretty much only one event can occur in a very small interval of time. While the probability is not exactly zero for more than one event, it is negligible.

2. The above conditions imply that

   $$P(X(t+h)-X(t)=0|X(t)=i)=1-(\lambda_i+\mu_i)h+o(h).$$

Example:

- Suppose that we have a population in which individuals do not reproduce.
- Suppose that the death rate per individual is μ. (This formally means that the probability that an individual dies in a really small time interval of length h is μh. Technically, it is $\mu h+o(h)$, but I'm going to stop dragging around some of that negligable stuff because it's all going to pile up in one "big fat little 'oh' of h term" anyway.)
- Suppose that new individuals immigrate into the population according to a Poisson process with rate λ.
- Let $X(t)$ be the population size at time t.

Claim: This is a birth-and-death process.

Let's verify this and find the birth and death parameters.

Given that there are currently i people in the population, as $h \searrow 0$, the probability that we see a population increase of size 1 is given by

$$P(X(t+h) - X(t) = 1 | X(t) = i) = P(\text{one arrival, no deaths in interval of length } h) + o(h)$$

The $o(h)$ term represents that there are 2 arrivals and 1 deaths, 3 arrivals and 2 deaths, etc... For the Poisson arrival process, there is only a $o(h)$ probability of 2,3, etc... arrivals. In other words, as h gets really small, the chance that these scenarios occur is shrinking down to zero quickly.

Although it wasn't specifically mentioned in the set-up of the problem, the arrival process and the death processes are independent. We can assume this because otherwise we would have to have been given information about how they are related.

Since arrivals follow a Poisson process with rate λ, the probability of exactly one arrival in a time interval of length h is

$$\frac{e^{-\lambda h}(\lambda h)^1}{1!} = e^{\lambda h}(\lambda h).$$

Since there are i individuals in the population, each with probability μh of dying in a small time interval of length h, the probability that no one dies is the probability of getting a zero for a binomial random variable that counts the number of deaths out of i people:

$$\binom{i}{0}(\mu h)^0(1-\mu h)^{i-0} = (1-\mu h)^i.$$

Putting this all together, we have

$$\begin{aligned}
P(X(t+h) - X(t) = 1 | X(t) = i) &= P(\text{one arrival, no deaths in interval of length } h) + o(h) \\
&= e^{-\lambda h}(\lambda h) \cdot (1-\mu h)^i + o(h) \\
&= \left[1 - \lambda h + \tfrac{\lambda^2 h^2}{2!} - \tfrac{\lambda^3 h^3}{3!} + \cdots\right](\lambda h) \cdot \textstyle\sum_{k=0}^{i} \binom{i}{k} (1)^k (-\mu h)^{i-k} + o(h) \\
&= \left[\lambda h - \lambda^2 h^2 + \tfrac{\lambda^3 h^3}{2!} - \tfrac{\lambda^4 h^4}{3!} + \cdots\right] \cdot [1 - i\mu h + o(h)] + o(h) \\
&= (\lambda h + o(h))(1 - i\mu h + o(h)) + o(h) \\
&= \lambda h + o(h)
\end{aligned}$$

Groovy.

Now

$$\begin{aligned}
P(X(t+h)-X(t)=-1|X(t)=i) &= P(\text{one death, no arrivals in interval of length } h)+o(h) \\
&= i\cdot\mu h(1-\mu h)^{i-1}\cdot e^{-\lambda h}+o(h) \\
&= i\mu h[1-(i-1)\mu h+o(h)]\cdot\left[1-\lambda h+\tfrac{\lambda^2h^2}{2!}-\cdots\right]+o(h)
\end{aligned}$$

The $[1-(i-1)\mu h+o(h)]$ term comes from a similar binomial expansion computation of $(1-\mu h)^{i-1}$ as was done above for $(1-\mu h)$.

Continuing... Now

$$\begin{aligned}
P(X(t+h)-X(t)=-1|X(t)=i) &= P(\text{one death, no arrivals in interval of length } h)+o(h) \\
&= i\cdot\mu h(1-\mu h)^{i-1}\cdot e^{-\lambda h}+o(h) \\
&= i\mu h[1-(i-1)\mu h+o(h)]\cdot\left[1-\lambda h+\tfrac{\lambda^2h^2}{2!}-\cdots\right]+o(h) \\
&= [i\mu h+o(h)]\cdot[1-\lambda h+o(h)] \\
&= i\mu h+o(h)
\end{aligned}$$

So, this is looking like a birth and death process with

$$\lambda_i=\lambda, \qquad \text{and} \qquad \mu_i=i\mu$$

for $i=1,2\ldots$.

We still need to check one of the two other possibilities for $X(t+h)-X(t)$. $(=0$ or $\geq 2)$ Then, the final case will hold automatically. I'll check the zero case:

$$P(X(t+h)-X(t)=0|X(t)=i)=P(\text{no arrival, no death})+P(\text{1 arrival, 1 death})+o(h)$$

Wait! Can we have two events (1 arrival, 1 death) happen at the same small instant of time? The answer is no, but this really is a property assumed for a single Poisson process. We have a combination of birth and death events. We will now show formally that this second term is still $o(h)$.

$$
\begin{aligned}
P(X(t+h)-X(t)=0|X(t)=i) &= P(\text{no arrival, no death}) + P(\text{1 arrival, 1 death}) + o(h) \\
&= e^{-\lambda h}\binom{i}{0}(\mu h)^0(1-\mu h)^i + \lambda h e^{-\lambda h}\binom{i}{1}(\mu h)(1-\mu h)^{i-1} + o(h) \\
&= e^{-\lambda h}(1-\mu h)^i + \lambda\mu h^2 e^{-\lambda h}(1-\mu h)^{i-1} + o(h) \\
&= e^{-\lambda h}(1-\mu h)^i + o(h) \\
&= [1-\lambda h + o(h)]\cdot[1 - i\mu h + o(h)] + o(h) \\
&= 1-(\lambda + i\mu)h + o(h),
\end{aligned}
$$

as desired.

In all of the birth-and-death examples we have seen so far, we have probabilities written as

$$\text{“something”}\ h + o(h).$$

These expressions hold for very very small values of h ($h \searrow 0$). This is why they are sometimes called "infinitessimal transition rates". We will use q_{ij} to denote the rate of an infinitessimal transition from state i to state j. For example, for a birth process, we have

$$
p_{ij}(h) = \begin{cases} \lambda_i h + o(h), & \text{if } j = i+1 \\ 1 - \lambda_i h + o(h), & \text{if } j = i \\ o(h), \text{otherwise} \end{cases}
$$

We can further condense this notation by writing

$$p_{ij}(h) = \delta_{ij} + q_{ij}h + o(h)$$

where

$$
\delta_{ij} = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases}, \qquad q_{ij} = \begin{cases} -\lambda_i, & j = i \\ \lambda_i, & j = i+1 \\ 0, & \text{otherwise} \end{cases}
$$

Using the expression $p_{ij}(h) = \delta_{ij} + q_{ij}h + o(h)$ and the fact that $[o(h)/h] \searrow 0$ as $h \searrow 0$, we can write

$$q_{ij} = \lim_{h \searrow 0} \frac{p_{ij}(h) - \delta_{ij}}{h}$$

but we may write $\delta_{ij} = p_{ij}(0)$ because we go from i to i in zero timesteps with probability 1 and from i to $j \neq i$ in zero time steps with probability 0. So, we have

$$q_{ij} = \lim_{h \searrow 0} \frac{p_{ij}(h) - p_{ij}(0)}{h} = p'_{ij}(0)$$

where $p'_{ij}(0)$ is the derivative of $p_{ij}(t)$ with respect to t and evaluated at zero. (Technically, it's a "right derivative".)

So, it seems like we might be able to determine the transition matrix $\mathbf{P}(t)$ from knowledge of the matrix $\mathbf{P}'(0)$ since $\mathbf{P}$ is related to the q_{ij} which are in turn related to the $p'_{ij}(0)$. Hmmmm... read on...

Notation: Let

$$\mathbf{Q} = [q_{ij}] = [p'_{ij}(0)] = \mathbf{P}'(0).$$

Consider the Chapman-Kolmogorov equations:

$$p_{ij}(s+t) = \sum_k p_{ik}(s)p_{kj}(t).$$

Differentiating with respect to t gives us

$$p'_{ij}(s+t) = \sum_k p_{ik}(s)p'_{kj}(t).$$

Now setting $t = 0$ gives us

$$\begin{aligned} p'_{ij}(s) &= \textstyle\sum_k p_{ik}(s)p'_{kj}(0) \\ &= \textstyle\sum_k p_{ik}(s)q_{kj}. \end{aligned}$$

The left hand side is the ijth element of $\mathbf{P}(s)$. The righthand side is the sum of the elements in the ith column of $\mathbf{P}(s)$, each multiplied by the elements in the jth column of $\mathbf{Q}$. Therefore, we have

$$\boxed{\mathbf{P}'(s) = \mathbf{P}(s)\mathbf{Q}}.$$

This is know as the **Kolmogorov Forward Equation**.

Repeating the above procedure while switching the roles of s and t (ie: differentiate with respect to s and setting $s = 0$) give us the **Kolmogorov Backward Equation**:

$$\boxed{\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t)}.$$

I have an idea... let's try to solve the differential equation given by the Kolmogorov backward equation so that we have an explicit expression for $\mathbf{P}(t)$!

If these were not matrices, this would be pretty easy. If, we had a function $p(t)$ and a constant q such that

$$p'(t) = qp(t),$$

we would write

$$p'(t) = qp(t) \qquad \Rightarrow \qquad \frac{p'(t)}{p(t)} = q \qquad \Rightarrow \qquad \frac{d}{dt}\ln(p(t)) = q$$

... integrating both sides...

$$\Rightarrow \qquad \ln p(t) = qt + c \qquad \Rightarrow p(t) = ae^{qt}$$

where

$$a = e^c.$$

Inspired by this, we define the <u>exponential matrix</u>:

$$e^{Qt} := \sum_{n=0}^{\infty} Q^n \cdot \frac{t^n}{n!}.$$

Differentiating this gives us

$$\frac{d}{dt}e^{Qt} = \sum_{n=0}^{\infty} \mathbf{Q}^n \cdot \frac{t^{n-1}}{(n-1)!} = Q\sum_{n=1}^{\infty} Q^{n-1}\frac{t^{n-1}}{(n-1)!} = Qe^{Qt}$$

So, we see that

$$\boxed{\mathbf{P}(t) = e^{Qt}}$$

solves

$$\mathbf{P}'(t) = \mathbf{QP}(t).$$

That's pretty cool if you ask me... I guess you didn't ask though...