Handout - WEEK 10

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Martingales and the doubling bet strategy.

Suppose that you play a game against a Casino where you win or lose \$1 per each dollar you bet. The *doubling bet strategy* consists on the following:

(a) you start the game betting only \$1, and

(b) every time you loose you bet twice as much the amount you bet before, however,

(c) every time you win you bet only \$1 on the next game.

For example, suppose that the outcomes of the first five games turn out to be as summarized in the following table:

$n =$	1	2	3	4	5	6
bet for game n	\$1	\$2	\$4	\$8	\$16	\$1
profit/(dollar bet) on game n	-\$1	-\$1	-\$1	-\$1	\$1	?
net profit up to game n	-\$1	-\$3	-\$7	-\$15	\$1	?

Observe that a -\$1 profit per dollar bet means that you lost in the game. Instead, a \$1 profit per dollar bet means that you won. Furthermore, observe how by the end of the fifth game you have made a dollar of profit. Putting that dollar aside we can think that the game starts afresh on the sixth game.

More generally, if you start with a bet of $\$2^0$ and loose k-times in a row but on the $(k+1)$-th bet you win then, using the above strategy, your net profit by the end of the $(k+1)$-th game will be

$$-(2^0 + \ldots + 2^{k-1}) + 2^k = 2^k - \sum_{i=0}^{k-1} 2^i = 2^k - \frac{2^k - 1}{2 - 1} = \$1\,.$$

Since every time you win the game continues as if it started afresh, the doubling bet strategy ensures that you will became rich provided that you can always double your last bet and that there is a positive probability of winning in each game. This conclusion seems to contradict common wisdom. There has to be a catch or otherwise by now there would be no casinos left in Las Vegas. The catch is that the amount of time needed to make a \$1 profit is random and eventually you will not have enough money to double your last bet. For example, even if you had \$1,000,000 just to spend in this game in Las Vegas, you cannot afford to lose more than 19-times in a row (which is the integer part of $\log_2(1 + 10^6)$). Indeed, by the time you have lost 20-times in a row you better start running because you will own \$1,048,575 to the Casino i.e. you are about \$50,000 off.

To model the situation probabilistically we let W_n be the random net profit after the n-th game. We also let H_n be the amount we bet before the n-th game and define the random variables

$$X_n := \begin{cases} -1 & , \quad \text{if one looses on the } n\text{-th game;} \\ +1 & , \quad \text{if one wins on the } n\text{-th game.} \end{cases}$$

We can think of X_n as the profit per dollar bet on the n-th game. To clarify the notation observe that these random variables take the following values for the game summarized in the above table:

$n =$	1	2	3	4	5	6
$H_n =$	1	2	4	8	16	1
$X_n =$	-1	-1	-1	-1	+1	?
$W_n =$	-1	-3	-7	-15	1	?

If one defines $X_0 := 0$ and $W_0 := 0$ then it follows that:

(i) For all $n \geq 1$, $W_n = W_0 + \sum_{i=1}^{n} H_i \cdot X_i$. In particular, if we define $Y_n := \sum_{i=0}^{n} X_i$, it follows that

$$W_n = W_0 + \sum_{i=1}^{n} H_i \cdot (Y_i - Y_{i-1}) \, .$$

(ii) $Y_n \in \mathcal{F}(X_0, \ldots, X_n)$. Furthermore, since you are playing the game against a casino you can be assured that $E(Y_{n+1} \mid X_n, \ldots, X_0) < Y_n$. Indeed,

$$\begin{aligned} E(Y_{n+1} \mid X_n, \ldots, X_0) = E\left(\sum_{i=1}^{n+1} X_i \mid X_n, \ldots, X_0 \right) &= \sum_{i=1}^{n} X_n + E\left(X_{n+1} \mid X_n, \ldots, X_0\right) \\ &= Y_n + E(X_{n+1}) \, . \end{aligned}$$

Since the only way casinos can be profitable in the long-run is to promote games where $E(X_{n+1}) < 0$, the above implies that $E(Y_{n+1} \mid X_n, \ldots, X_0) < Y_n$ as claimed. This shows that $(Y_n)_{n \geq 1}$ is a supermartingale w.r.t. $(X_n)_{n \geq 1}$. In words this means that the game is unfair for gamblers.

(iii) Finally, observe that $H_n \in \mathcal{F}(X_0, \ldots, X_{n-1})$. This is because, your decision on how to bet on the n-th game (i.e. to just bet \$1 or doubling the previous bet) only uses information about the previous games. This information is completely conveyed in the random vector $(X_0, \ldots, X_{n-1})$.

Using (i)-(iii), theorem 3.2 implies that $(W_n)_{n \geq 0}$ is a supermartingale w.r.t. $(X_n)_{n \geq 1}$ i.e. for all $n \geq 0$

$$E(W_{n+1} \mid X_n, \ldots, X_0) \leq W_n \, .$$

Taking expected values both sides above we see that $E(E(W_{n+1} \mid X_n, \ldots, X_0)) \leq E(W_n)$ i.e. $E(W_{n+1}) \leq E(W_n)$. In words, even if one had no monetary restrictions to use the doubling bet strategy, the average net profit one would make playing an unfair game can only get worse the more the games played. In particular, since $W_0 = 0$, for all $n \geq 1$, $E(W_n) \leq E(W_0) = 0$. This insights prove the following result.

Theorem 3.1 (pp 111). *Suppose we use the doubling bet strategy on a game where you win or lose \$1 per each dollar you bet. If the game is unfair for gamblers then for all* $n \geq 0$, $E(W_{n+1}) \leq E(W_n) \leq 0$.