ANSWER KEY

APPM 4/5560: Markov processes, queues and Monte Carlo simulation - Fall 2008

Exam #1
Lecturer: Manuel Lladser

INSTRUCTIONS: On the front of your bluebook please print your *name*, *student ID*, *course code*, *exam number*, *date* and *lecturer's name*. Please draw a grading table (with *2 columns* and *5 rows*). Show all your work in your bluebook. Please start each new problem in a *new page*. Solve the problems in the *same order* as they are requested. A correct answer with no supporting work may receive no credit while and incorrect answer with some correct work may receive partial credit. *Textbooks, class notes, graphing or programmable calculators, and crib sheets are not permitted.*

1. (50 points.) Considere a time-homogeneous Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ and probability transition matrix

$$p = \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2/3 & 0 & 1/3 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \end{bmatrix}.$$

Follow the next instructions and respond. **You do not need to justify your answers.**

(a) Represent the probability transition matrix as a directed graph with weighted edges.
(b) What does the entry in row-3 and column-5 of p^{925} represent?
(c) What is the probability of the event $X_n = 1$ given that $X_0 = 1$?
(d) Which states are recurrent? Which are transient?
(e) Determine the period of each state.
(f) Is this chain irreducible?
(g) Give three different examples of a closed set of states.
(h) Does this chain have a unique stationary distribution? If so, specify it. Otherwise, provide at least two different stationary distributions.
(i) Intuitively, what should be the limit $\lim_{n\to\infty} p^n(1, 2)$? Explain very briefly your reasoning.
(j) Does the limit $\lim_{n\to\infty} p^n(3, 5)$ exist? If so, determine its value. Otherwise, explain why it does not exist.

2. (10 points.) Explain how you could simulate using a random variable $U \sim Uniform[0, 1]$ the toss of a coin which has probability $1/\sqrt{2}$ of coming up heads.

3. (10 points.) Show how using a random variable $U \sim Uniform[0, 1]$ you could simulate a random variable with cumulative distribution function given by the formula

$$F(x) = \begin{cases} 0 & , \quad x < 0; \\ \frac{\sqrt{1+4x}-1}{2} & , \quad 0 \le x \le 2; \\ 1 & , \quad x > 2. \end{cases}$$

(One more question on the back!)

4. (30 points.) Let X be a random variable with *moment generating function* given by the formula

$$\varphi_X(t) = E(e^{tX}) = \frac{e^{t+1} - 1}{(t+1) \cdot (e-1)}.$$

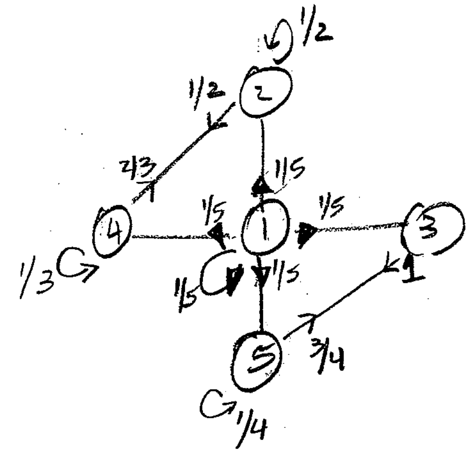
(a) Determine the expected value of X.

(b) Determine the moment generating function of $(-X)$.

(c) Do X and $(-X)$ have the same distribution? Explain.

DURATION: 50 MINUTES

P1



(b) $= P[X_{125} = 5 \mid X_0 = 3]$

(c) $= P[X_m = 1 \mid X_0 = 1] = P[X_m = 1, \ldots, X_1 = 1 \mid X_0 = 1]$

$= [p(1,1)]^m = \left(\frac{1}{5}\right)^m$

(d) : recurrent states are 2, 3, 4, 5.
1 is the only transient state

(e) : period (1) = ... = period (5) = 1.

(f) : NO

(g) : $\{3,5\}$, $\{2,4\}$, $\{1,2,3,4,5\}$.

(h) : NO. $\pi_1 = [0 \;\; 0 \;\; 3/7 \;\; 0 \;\; 4/7]$ is stationary if the chain starts at 3 or 5 i.e.

$$[3/7 \;\; 4/7] \begin{bmatrix} 0 & 1 \\ 3/4 & 1/4 \end{bmatrix} = [3/7 \;\; 4/7]$$

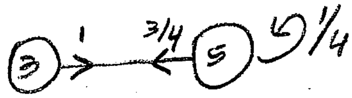
and

$\pi_2 = [0 \;\; 4/7 \;\; 0 \;\; 3/7 \;\; 0]$ is stationary if the chain starts at 2 or 4 i.e.

$$[4/7 \;\; 3/7] \begin{bmatrix} 1/2 & 1/2 \\ 2/3 & 1/3 \end{bmatrix} = [4/7 \;\; 3/7]$$

$$\text{(i)} \overset{\text{symmetry}}{=} \frac{1}{2}\cdot \pi_2(2) = \frac{1}{2}\cdot\frac{4}{7} = \frac{2}{7}.$$

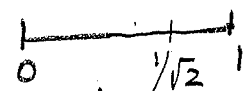
(j): Yes because $\{3,5\}$ is a closed set of states and the chain has transition probabilities



which is an irreducible an aperiodic chain. Hence

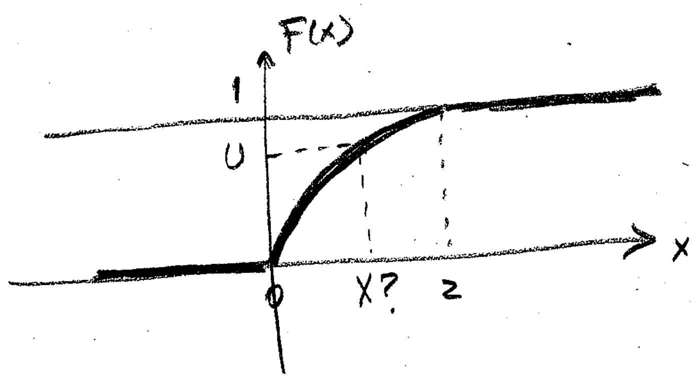
$$\lim_{n\to\infty} p^n(3,5) = \pi_1(5) = \frac{4}{7}.$$

P2 Split $[0,1]$ into two intervals:



If $U < 1/\sqrt{2}$ say the coin came up heads.
Otherwise, say the coin came up tails.

P3.



Need to find $F^{-1}(U) = X$ i.e. solve

$$\frac{\sqrt{1+4X}-1}{2} = U \quad \text{with } 0 \le X \le 2$$

$$1+4X = (2U+1)^2$$

$$X = \frac{(2U+1)^2-1}{4} = U^2+U \in [0,2].$$

So (U^2+U), with $U \sim \text{Uniform}[0,1]$ has c.d.f. $F(x)$

P4

(a) $= \varphi_X'(0)$. Since

$$\varphi_X'(t) = \frac{1}{e-1} \cdot \frac{e^{t+1}(t+1) - (e^{t+1}-1)\cdot 1}{(t+1)^2} = \frac{te^{t+1}+1}{(e-1)(t+1)^2}$$

we obtain $E(X) = \frac{1}{e-1}$.

(b) $= \varphi_{(-X)}(t) = E(e^{-tX}) = \varphi_X(-t) = \frac{e^{1-t}-1}{(1-t)\cdot(e-1)}$

(c) : clearly $\varphi_X(t) = \frac{e^{t+1}-1}{(t+1)(e-1)} \neq \frac{e^{1-t}-1}{(1-t)\cdot(e-1)} = \varphi_{(-X)}(t)$

for at least one value of t, hence X and $(-X)$ cannot have the same distribution $\#$

Note: In part (c), you could use that $E(X) \neq 0$ to conclude that $E(-X) = -E(X) \neq E(X)$ hence X and $(-X)$ cannot have the same distribution.