
APPM 4/5560: Markov processes, queues and Monte Carlo simulation - Fall 2008

Exam #2

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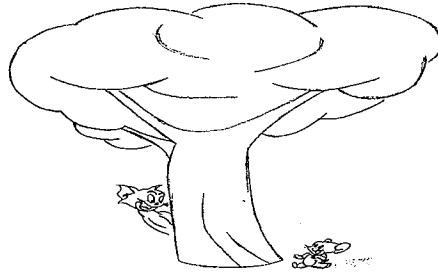
INSTRUCTIONS: On the front of your bluebook please print your *name, student ID, course code, exam number, date* and *lecturer's name*. Please draw a grading table (with 2 columns and 5 rows). Show all your work in your bluebook. Please start each new problem in a *new page*. Solve the problems in the *same order* as they are requested. A correct answer with no supporting work may receive no credit while an incorrect answer with some correct work may receive partial credit. *Textbooks, class notes, graphing or programmable calculators, and crib sheets are not permitted.*

1. (60 points.) In what follows $(N_t)_{t \geq 0}$ represents a homogenous Poisson process with intensity $\lambda = \ln(2)$ and arrival times $T_1 < T_2 < \dots$. Decide whether each of the following statements is TRUE (i.e. always true) or FALSE (i.e. not always true). **Do not justify your answers.**
 - (a) $P[T_1 \geq 2] = 1/4$.
 - (b) For each $k \geq 1$, T_k has density function $P[T_k = t] = k \cdot \ln(2) \cdot 2^{-t}$, for $t \geq 0$.
 - (c) If $N_5 = 3$ and $N_7 = 3$ then $T_4 > 7$.
 - (d) If $T_5 < 9$ then $N_9 \geq 5$.
 - (e) $P[N_5 = 0] = 1/\ln(32)$.
 - (f) The expected number of arrivals in the time interval $[0, 5]$ is 32.
 - (g) $E(N_9 - N_5) = E(N_{12} - N_8)$.
 - (h) The random variable $(T_9 - T_5)$ has mean $4/\ln(2)$.
 - (i) Conditioned on the event $T_4 < 5 < T_5$, the event $[(T_5 - 5) \geq 3]$ has probability $1/8$.
 - (j) Conditioned on the event $T_4 < 5 < T_5$, the random variable $(T_9 - 5)$ has a Gamma distribution with mean $5/\ln(2)$.
 - (k) $P[N_9 - N_5 = 3 \mid N_{10} = 8] = \left(\frac{2}{5}\right)^3 \cdot \left(\frac{3}{5}\right)^5$.
 - (l) $P[T_8 \leq x \mid N_{10} = 8] = \left(\frac{x}{10}\right)^8$, for $0 \leq x \leq 10$.

(Two more problems on the back!)

2. (20 points.) Tom the cat is trying to catch Jerry the mouse who is hiding behind a tree (see figure below). Please respond to the next questions based on the following assumptions:

- at any given moment Tom and Jerry may only be on the right-hand side (R) or the left-hand side (L) of the tree,
- Tom stays at its current position with probability $1/3$, or switches to the opposite side of the tree with probability $2/3$ (independently of Jerry),
- Jerry is equally likely to stay at its current position or to switch to the opposite side of the tree (independently of Tom).



- (a) Define the probability transition matrix of a Markov chain with state space $\{(R, L), (L, R), (R, R), (L, L)\}$ to model the above situation. Represent this matrix as a directed graph with weighted edges.
- (b) If $X_0 = (R, L)$, how many moves does it take on average for Tom to catch Jerry?

Please solve only one of the following problems. The average score among problems 3 and 4 will be considered if you solve both of them!

3. (20 points.) Imagine a flight attendant in charge of 12 passengers in first-class in a 15 hours flight from Washington D.C. to Johannesburg in South Africa. If the pilot of the plane will contact the attendant on average every 30 minutes and, each of the first-class passengers will request assistance on average every 2 hours, please answer:
- (a) How would you model the times at which the attendant is contacted by the pilot or the first-class passengers? Explain.
- (b) On average, which fraction of the contacts along the duration of the flight will be due to the first-class passengers?
4. (20 points.) Imagine a backpacker who waits at a bus-stop to get back to Salt Lake City (SLC) after having visited the Grand Canyon. Suppose that a bus passes by the stop towards SLC every h hours; in particular, he will have to wait for the next bus an amount of time that is uniformly distributed between 0 and h hours. Assuming that cars pass by the bus-stop at a rate of 2 per hour but only 1 in 5 cars will take the backpacker to SLC, please answer:
- (a) If the next bus were to pass exactly t hours after the backpacker arrived to the bus-stop, what is the probability that the backpacker would still be waiting?
- (b) What is the probability that the backpacker will end up taking the bus to SLC?

DURATION: 50 MINUTES

Answer key - Exam #2 - APPM 4/5560 - Fall 2008

171 a: TRUE b/c $P[T_1 \geq 2] = e^{-\lambda \cdot 2} = 1/4$

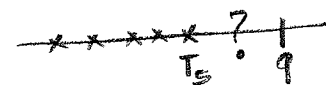
b: FALSE b/c $\int_0^{\infty} k \ln(2) 2^{-t} dt = k \int_0^{\infty} \ln(2) \cdot e^{-\ln(2)t} dt = k > 1$
when $k \geq 2$.

c: TRUE b/c



A horizontal line representing time with three tick marks labeled 5, 7, and T_4 . Above the line, there are three 'x' marks corresponding to these times.

d: TRUE b/c



A horizontal line representing time with two tick marks labeled T_5 and 9. Above the line, there are five 'x' marks between T_5 and 9, and one 'x' mark at 9.

e: FALSE b/c $P[N_5 = 0] = P[P_0(5 \cdot \ln(2)) = 0] = e^{-5 \ln 2} = 1/32$

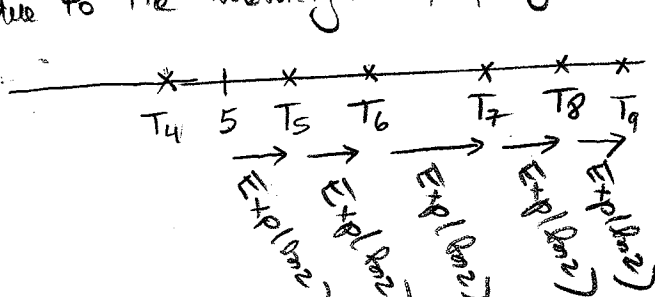
f: FALSE b/c $E(N_5) = 5 \cdot \ln(2) = \ln(32)$

g: TRUE b/c $(N_9 - N_5)$ and $(N_{12} - N_8)$ are both Poisson $(4 \cdot \ln(2))$

h: TRUE b/c $E(T_9 - T_5) = \sum_{i=6}^9 E(T_i - T_{i-1}) = \sum_{i=6}^9 \frac{1}{\ln(2)} = \frac{4}{\ln(2)}$

i: TRUE b/c $P[T_5 - 5 \geq 3 | T_4 < 5 < T_5] = P[T_1 \geq 3] = e^{-3 \ln(2)} = 1/8$
due to the memoryless property

j: TRUE b/c



A horizontal line representing time with seven tick marks labeled T_4 , 5, T_5 , T_6 , T_7 , T_8 , and T_9 . Above the line, there are seven 'x' marks corresponding to these times. Below the line, arrows point to the intervals between consecutive marks, each labeled with an exponential distribution: $Exp(\ln(2))$, $Exp(\ln(2))$, $Exp(\ln(2))$, $Exp(\ln(2))$, $Exp(\ln(2))$, and $Exp(\ln(2))$.

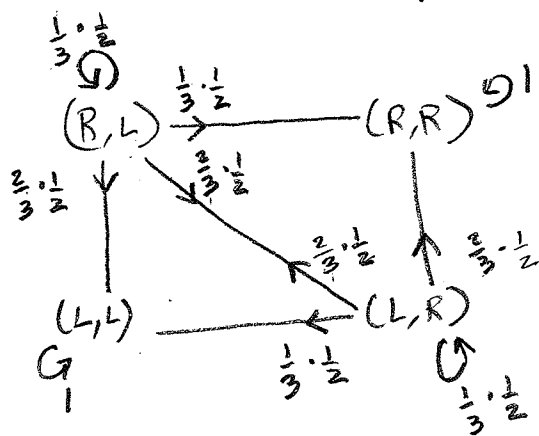
hence $(T_9 - 5)$ has a Gamma $(5, \ln(2))$ distribution.

k: FALSE b/c $P[N_9 - N_5 = 3 | N_{10} = 8] = \binom{8}{3} \left(\frac{4}{10}\right)^3 \left(\frac{6}{10}\right)^5$

l: TRUE b/c $P[T_8 \leq x | N_{10} = 8] = P[\max_{i=1 \dots 8} U_i \leq x] = \left(\frac{x}{10}\right)^8$,
where U_1, \dots, U_8 are i.i.d. Uniform $[0, 10]$.

P2

(a) We will (T, J) represent the position of Tom (T) and Jerry (J).



(b) Let $T = \#$ steps until T catches J.
 Due to the symmetry, the expected value of T is the same if $X_0 = (R, L)$ or $X_0 = (L, R)$. Hence, assuming that $X_0 = (R, L)$ we have

$$E(T) = \frac{1}{6} \{1 + E(T)\} + \frac{1}{6} \cdot 1 + \frac{2}{6} \cdot 1 + \frac{2}{6} \cdot \{1 + E(T)\}$$

$$E(T) = 1 + \left\{ \frac{1}{6} + \frac{2}{6} \right\} E(T)$$

$$\frac{1}{2} E(T) = 1, \text{ hence } \boxed{E(T) = 2}$$

P3

(a) We may model the situation as a superposition of a HPP w/intensity $\lambda_{\text{Pilot}} = 2$ and fifteen HPP's each w/intensity $\lambda_{\text{single-passenger}} = \frac{1}{2}$. Assuming all these processes are indep., the times at which the attendant will be contacted follow a HPP w/intensity $\lambda = 2 + 15 \cdot \frac{1}{2} = 8$.

(b)
$$= \frac{12 \cdot \frac{1}{2}}{2 + 12 \cdot \frac{1}{2}} = \frac{6}{8} = \boxed{\frac{3}{4}}$$

P4

(a) The times at which cars would take the backpacker cars can be modelled as a HPP w/ intensity $2 \cdot 1/5 = 2/5$ per hour.

Here

$$\text{ANS} = P[\text{Poisson}(t/5) = 0] = e^{-2t/5}$$

$$(b) = \int_0^h e^{-2t/5} \cdot P[\text{Bus passes at time } t] dt$$

$$= \int_0^h e^{-2t/5} \cdot \frac{1}{h} dt$$

$$= \frac{1}{h} \left(-\frac{5}{2} e^{-2t/5} \Big|_{t=0}^{t=h} \right)$$

$$= \boxed{\frac{5(1 - e^{-2h/5})}{2h}}$$