
APPM 4/5560: Markov processes, queues, and simulation - Fall 2006

Exam 2 - Markov chains and Homogenous Poisson Processes, Sections 1.3 - 3.4

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INSTRUCTIONS:

On the front of your bluebook please print your **name**, **course name** and **term**, **date**, **lecturer's name** and draw a grading table (with **2 columns** and **5 rows**). Show all your work in your bluebook and **box in your final answers** where appropriate. Please **start each new problem in a new page**. A correct answer with no supporting work may receive no credit while and incorrect answer with some correct work may receive partial credit. **Textbooks, class notes, graphing or programmable calculators, and crib sheets are not permitted.**

P1. (25 points.) Let $X = (X_n)_{n \geq 0}$ be a first-order homogeneous Markov chain with state space $\{1, 2, 3\}$ and probability transition matrix

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \end{bmatrix}.$$

(a) Determine the stationary distribution of X.

(b) Does the limit

$$\lim_{n \to \infty} P[X_{2n} = 3, X_n = 2 \,|\, X_{n-1} = 2, X_{n-2} = 1]$$

exist? If so, what's its value?

(c) Does the limit

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i^2$$

exist? If so, what's its value?

P2. (25 points.) Consider a population in which individuals in any generation give birth to an independent and identically distributed number of children. Denote by X_n the number of individuals in generation n. We can model the generation sizes by defining recursively $X_0 := 1$ and

$$X_{n+1} = \sum_{j=1}^{X_n} Z_{j,n},$$

where $Z_{j,n}$ represents the number of offsprings of individual j in generation n. Due to modeling assumptions observe that the random variables $Z_{j,n}$ must be i.i.d. Let $Z = Z_{1,0}$ and suppose that $Z \sim \mathit{Binomial}(n = 2, p)$ where $0 < p < 1$ is a parameter; in particular,

$$P[Z = k] = \binom{2}{k} p^k (1-p)^{2-k}, \qquad k = 0, 1, 2.$$

(a) Briefly explain why the condition $2p < 1$ implies that the population goes extinct with probability one.

(b) Determine the generating function $\varphi(\theta) = E(\theta^Z)$.

Hint: For all integer $n \geq 0$ and $a, b \in \mathbb{R}$, $\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$.

(c) If $p = 3/4$, what's the probability that the population goes extinct conditioning on the event $X_7 = 3$?

(two more problems on the back)

P3. (25 points.) Two gangsters A and B suddenly meet at a corner of a street in downtown Chicago and start shooting each other.

Let $\alpha \in (0,1)$ be the probability that A hits B.

Similarly, let $\beta \in (0,1)$ be the probability that B hits A.

Assume that the gangsters shoot each other simultaneously and do so until one or both go down. Consider the Markov chain $X = (X_n)_{n \geq 0}$ with state space $\{(A,B), (\dagger, B), (A, \dagger), (\dagger, \dagger)\}$, where $X_0 = (A,B)$ and for $n \geq 1$,

$$\begin{aligned} X_n &= (A,B) \text{ if right after the } n\text{-th shoot } A \text{ and } B \text{ are alive;} \\ X_n &= (\dagger, B) \text{ if right after the } n\text{-th shoot } A \text{ is dead and } B \text{ is alive;} \\ X_n &= (A, \dagger) \text{ if right after the } n\text{-th shoot } A \text{ is alive and } B \text{ is dead;} \\ X_n &= (\dagger, \dagger) \text{ if right after the } n\text{-th shoot } A \text{ and } B \text{ are dead.} \end{aligned}$$

(a) Represent the probability transition matrix of X as a graph with weighted edges.

(b) What's the probability that A survives?

P4. (25 points.) Let $\lambda, \mu > 0$ be certain parameters. Suppose that $S \sim Exp(\lambda)$, $T \sim Exp(\mu)$ and that S and T are independent random variables. In this problem you will be given instructions to determine the density function of $\max\{S,T\} - \min\{S,T\}$.

(a) For $0 < x < y$ determine the joint densities $P(S = x, T - S = y)$ and $P(T = x, S - T = y)$.

(b) Rewrite the event $[\min\{S,T\} = x, \max\{S,T\} - \min\{S,T\} = y]$ in terms of the events of part (a). Use this to determine the joint density function $P(\min\{S,T\} = x, \max\{S,T\} - \min\{S,T\} = y)$.

(c) Use the above to obtain the p.d.f. of the random variable: $\max\{S,T\} - \min\{S,T\}$.

Duration: 50 minutes

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P1 ⓐ $\pi_1 \cdot \frac{1}{2} + \pi_3 \cdot \frac{1}{2} = \pi_1 \iff \boxed{\pi_1 = \pi_3}$

$\pi_1 \cdot \frac{1}{2} + \pi_2 \cdot \frac{2}{3} = \pi_2 \iff \boxed{\pi_2 = \frac{3}{2}\pi_1}$

Since $\pi_1 + \pi_2 + \pi_3 = 1$ then $\pi_1 \cdot \frac{7}{2} = 1$ hence $\boxed{\pi_1 = \frac{2}{7}}$, $\boxed{\pi_2 = \frac{3}{7}}$ and $\boxed{\pi_3 = \frac{2}{7}}$

Since the chain is irreducible in a finite state space then a stationary distribution exists and is unique. Hence it is given by the above #

ⓑ $$= \lim_{n\to\infty} P[X_{2n} = 3, X_n = 2 \mid X_{n-1} = 2]$$

$$= \lim_{n\to\infty} P[X_{2n} = 3 \mid X_n = 2] \cdot P[X_n = 2 \mid X_{n-1} = 2]$$

$$\overset{\text{homogeneous chain}}{=} \lim_{n\to\infty} P[X_n = 3 \mid X_0 = 2] \cdot p(2,2)$$

$$\overset{\text{irreducible and aperiodic}}{=} \pi(3) \cdot p(2,2) = \frac{2}{7} \cdot \frac{2}{3} = \boxed{\frac{4}{21}}$$ #

ⓒ Since the chain is irreducible with stationary distribution π then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i^2 = 1^2 \cdot \pi_1 + 2^2 \cdot \pi_2 + 3^2 \cdot \pi_3$$

$$= \frac{2}{7} + \frac{4 \cdot 3}{7} + \frac{9 \cdot 2}{7} = \frac{2 + 12 + 18}{7} = \boxed{\frac{32}{7}}$$ #

P2 ⓐ Since $E(Z) = n \cdot p = 2p$, we know that $P[\text{extinction}] = 1$ whenever $2p < 1$.

ⓑ $$\varphi(\theta) = \sum_{k=0}^{2} P[Z = k] \cdot \theta^k = \sum_{k=0}^{2} \binom{2}{k} (\theta p)^k (1-p)^{2-k} \overset{\text{Hint}}{=} (\theta p + 1 - p)^2$$

ⓒ If $p = 3/4$ then $\varphi(\theta) = \frac{(1+3\theta)^2}{16}$. Solving $\varphi(\theta) = \theta$, $\theta \in [0,1]$

$(1+3\theta)^2 = 16\theta$

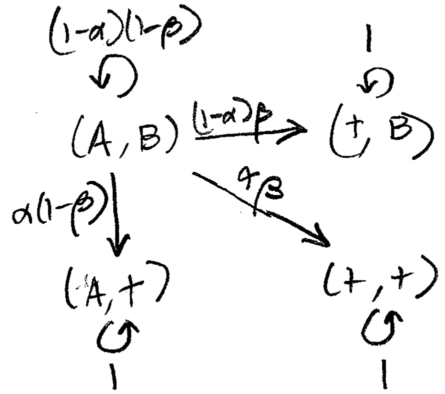
$9\theta^2 - 10\theta + 1 = 0$

hence
$$\theta = \frac{10 \pm \sqrt{100-36}}{18} = \frac{10 \pm 8}{18} \begin{cases} \theta = 1 \\ \theta = \frac{2}{18} = \frac{1}{9} \end{cases}$$

The probability of extinction starting with one individual is therefore $\rho = \frac{1}{9}$.

Hence $P[\text{extinction} \mid X_7 = 3] = \rho^3 = \boxed{\frac{1}{729}}$ because extinction can occur only if the descendants of each of the three individuals in generation 7 go extinct.

P3 ⓐ



ⓑ Let p be the probability that A survives. Since $X_0 = (A, B)$ then conditioning on X_1 we have that

$$p = (1-\alpha)(1-\beta)p + \overset{0}{\cancel{(1-\alpha)\beta \cdot 0}} + \alpha \cdot (1-\beta) \cdot 1 + \overset{0}{\cancel{\alpha \cdot \beta \cdot 0}}$$

$$[1-(1-\alpha)(1-\beta)]\,p = \alpha(1-\beta)$$

hence
$$p = \frac{\alpha(1-\beta)}{1-(1-\alpha)(1-\beta)} = \boxed{\frac{\alpha(1-\beta)}{\alpha+\beta-\alpha\beta}}$$

P4 ⓐ
$$P[S=x, T-S=y] = P[S=x, T=x+y] \overset{\text{indep}}{=} \lambda e^{-\lambda x} \mu e^{-\lambda(x+y)} = \boxed{\lambda\mu e^{-(\lambda+\mu)x} e^{-\lambda y}}$$

Similarly,
$$P[T=x, S-T=y] = \boxed{\lambda\mu e^{-(\lambda+\mu)x} e^{-\mu y}}$$

ⓑ
$$= P[S=x, T-S=y] + P[T=x, S-T=y] \overset{(a)}{=} \lambda\mu e^{-(\lambda+\mu)x}\left[e^{-\lambda y} + e^{-\mu y}\right]$$

ⓒ
$$= P[\max-\min = y] = \int_0^\infty P[\min = x, \max - \min = y]\,dx$$

$$= \int_0^\infty \lambda\mu e^{-(\lambda+\mu)x}\left[e^{-\lambda y}+e^{-\mu y}\right]dx = \frac{\lambda\mu}{\lambda+\mu}\left[e^{-\lambda y}+e^{-\mu y}\right] \cdot \underbrace{\int_0^\infty (\lambda+\mu)e^{-(\lambda+\mu)x}\,dx}_{1 \text{ because of density of an } Exp(\lambda+\mu).}$$

$$= \boxed{\frac{\lambda\mu\left[e^{-\lambda y}+e^{-\mu y}\right]}{\lambda+\mu}}$$