ANSWER KEY

APPM 4/5560: Markov processes, queues and Monte Carlo simulation - Fall 2008

Exam #3
Lecturer: Manuel Lladser

INSTRUCTIONS: On the front of your bluebook please print your *name*, *student ID*, *course code*, *exam number*, *date* and *lecturer's name*. Please draw a grading table (with *2 columns* and *4 rows*). Show all your work in your bluebook. Please start each new problem in a *new page*. Solve the problems in the *same order* as they are requested. A correct answer with no supporting work may receive no credit while and incorrect answer with some correct work may receive partial credit. *Textbooks, class notes, graphing or programmable calculators, and crib sheets are not permitted.*

1. (25 points.) Customers arrive at a single server at rate $\lambda > 0$ and require an exponential amount of time of service with rate $\mu > 0$. Customers waiting in line are impatient and if they are not in service they will leave at rate μ independent of their position in the queue. Assuming that all rates are per hour, please respond:

   (a) Let X_t be the total number of customers in the queue at time t. Formulate a Markov chain model for $(X_t)_{t\geq 0}$, representing the transition rates in a directed graph with weighted edges.

   (b) Is X_t a birth-death chain? Do not justify your answer.

   (c) Show that $(X_t)_{t\geq 0}$ has a stationary distribution and determine it explicitly.

   (d) On average, how many customers leave the queue after they receive service in an 8 hour period? (Do not count impatient customers who leave before they receive service!)

2. (50 points.) Consider two realtors, A and B, who share an office. The customers of A arrive at a rate of $\lambda_A > 0$ per hour while the customers of B arrive at a rate of $\lambda_B > 0$ per hour. Assuming that:

   - customers arrive on their own to the office,
   - the office has a maximum capacity of K customers (including those busy with the realtors),
   - each realtor spends an exponential amount of time with parameter μ with each customer,

   please respond:

   (a) Let X_t^A (resp. X_t^B) be the number of customers in the queue i.e. waiting or receiving service with realtor A (resp. B). Formulate a Markov chain model for (X_t^A, X_t^B) with states of the form (m, n), with $0 \leq (m+n) \leq K$, where m (resp. n) represents the customers in the queue of realtor A (resp. B).

   (b) Show using a mass conservation argument that

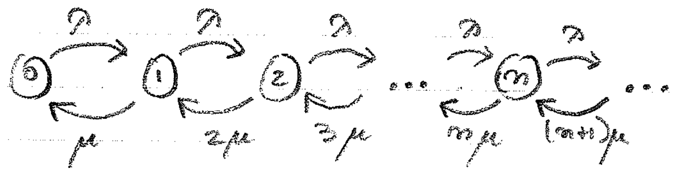
   $$\pi(m,n) = c \cdot \frac{\lambda_A^m \cdot \lambda_B^n}{\mu^{m+n}}$$

   is a stationary distribution for this chain, where $c > 0$ is an appropriate constant. You do not need to determine c explicitly.

   (c) Is this Markov chain time reversible? Explain briefly.

P1

ⓐ



ⓑ **YES!!**

ⓒ $\pi(n) = c \cdot \dfrac{\lambda^n}{\prod_{i=1}^{n}(i\mu)} = \boxed{c \cdot \dfrac{(\lambda/\mu)^n}{n!}}$.

Since

$$\sum_{n=0}^{\infty} \pi(n) = c \sum_{n=0}^{\infty} \frac{(\lambda/\mu)^n}{n!} = c \cdot e^{\lambda/\mu}, \text{ we need } \boxed{c = e^{-\lambda/\mu}}$$

ⓓ In an 8 hour period, the teller of the queue will be busy $8 \cdot (1 - \pi(0))$ hours. The average service time per customer is $1/\mu$, hence

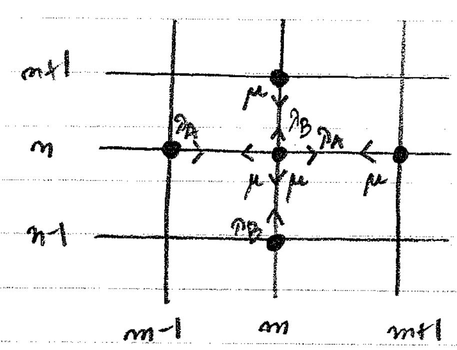
$$ANS = \frac{8 \cdot (1 - \pi(0))}{1/\mu} = 8\mu(1 - \pi(0))$$

P2

ⓐ

$$\begin{aligned}
q((m,n),(m-1,n)) &= \mu, && \text{for } 0 \le m+n \le K,\ m \ge 1 \\
q((m,n),(m,n-1)) &= \mu, && \text{for } 0 \le m+n \le K,\ n \ge 1 \\
q((m,n),(m+1,n)) &= \lambda_A, && \text{for } 0 \le m+n+1 \le K \\
q((m,n),(m,n+1)) &= \lambda_B, && \text{for } 0 \le m+n+1 \le K
\end{aligned}$$

ⓑ



Mass is conserved along each of the edges connected to (m,n)

n+1 / n :
$$\lambda_B \cdot \pi(m,n) = c\,\frac{\lambda_A^m \lambda_B^{n+1}}{\mu^{m+n}} = c\,\frac{\lambda_A^m \lambda_B^{n+1}}{\mu^{m+n+1}} \cdot \mu = \pi(m,n+1) \cdot \mu$$

n / n-1 :
$$\mu \cdot \pi(m,n) = c\,\frac{\lambda_A^m \lambda_B^n}{\mu^{m+n-1}} = c\,\frac{\lambda_A^m \lambda_B^{n-1}}{\mu^{m+n-1}} \cdot \lambda_B = \pi(m,n-1) \cdot \lambda_B$$

$(m-1)$ ●—● (m) : $\mu\, \pi(m,n) = c \dfrac{\lambda_A^{m} \lambda_B^{n}}{\mu^{m+n-1}} = c \dfrac{\lambda_A^{m-1} \lambda_B^{n}}{\mu^{m+n-1}} \cdot \lambda_A = \pi(m-1,n)\cdot \lambda_A$

(m) ●—● $(m+1)$: $\lambda_A\, \pi(m,n) = c \dfrac{\lambda_A^{m+1} \lambda_B^{n}}{\mu^{m+n}} = c \dfrac{\lambda_A^{m+1} \lambda_B^{n}}{\mu^{m+n+1}} \cdot \mu = \pi(m+1,n)\cdot \mu$

(c) **Yes !!** Part (a) shows that π satisfies the detailed balance condition

(d) $= \boxed{\sum_{m=1}^{K} \pi(m,0)}$

(e) $= \boxed{(\lambda_A + \lambda_B)\cdot \left\{1 - \sum_{m=0}^{K} \pi(m, K-m)\right\}}$

P3 (a)

$$r_1 = \lambda_1 + p(2,1)\cdot r_2$$
$$r_2 = p(1,2)\cdot r_1$$
$$r_3 = \lambda_3 + p(1,3)\cdot r_1 + p(2,3)\cdot r_2$$

(b) A customer starting at any queue will eventually reach queue-2 or queue-3 where it has a positive chance to leave.

(c)

$$r_2 = p(1,2)\cdot r_1$$
$$r_2 = p(1,2)\cdot \{\lambda_1 + p(2,1)\cdot r_2\}$$
$$r_2 - p(1,2)\, p(2,1)\, r_2 = p(1,2)\cdot \lambda_1$$

hence

$$\boxed{r_2 = \frac{\lambda_1\, p(1,2)}{1 - p(1,2)\, p(2,1)}}$$