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# APPM 4/5560: Markov processes, queues, and simulation - Fall 2006

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## Exam 3: Sections 3.3-4.6

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**INSTRUCTIONS:** On the front of your bluebook please print your **name, course name and term, date, lecturer's name** and draw a grading table (with **2 columns and 5 rows**). Show all your work in your bluebook and **box in your final answers** where appropriate. Please **start each new problem in a new page**. A correct answer with no supporting work may receive no credit while an incorrect answer with some correct work may receive partial credit. **Textbooks, class notes, graphing or programmable calculators, and crib sheets are not permitted.**

- P1.** (25 points.) Let  $N = (N_t)_{t \geq 0}$  be a homogenous Poisson process with intensity  $\lambda > 0$ .
- (a) If  $N_2 = 5$ , what's the probability that  $N_7 - N_5 = 2$ ? Simplify your answer.
  - (b) If  $N_9 = 5$ , what's the probability that  $N_7 - N_5 = 2$ ? Do not simplify your answer.
- P2.** (25 points.) A student waits for the downtown bus at the bus stop. The time until the next bus arrives is exponentially distributed with mean  $1/4$  of an hour. If cars drive by the bus stop at a rate of 6 per hour and each car will give him a ride to downtown with probability  $1/6$ , what's the probability that the student ends up taking the bus? Leave your answer as a reduced fraction.
- P3.** (25 points.) Suppose that calls are received at a 9-1-1 dispatch office at a rate of  $\lambda$  calls per hour. The office consists of  $m$  receptionists and the telephone system is set up so that a new call is forwarded with equal probability to any of the free receptionists (provided that at least one is free), otherwise the call is essentially lost as it is forwarded to a different emergency dispatch. If each receptionist spends an exponential amount of time with rate  $\mu$  with each call, respond:
- (a) Determine (but do not justify) the jump-rates of the Markov chain  $X = (X_t)_{t \geq 0}$  with state space  $\{0, \dots, m\}$ , where the state indicates the number of receptionists busy on the phone.
  - (b) Approximately, which fraction of the time are all the receptionists unoccupied? Leave your final answer in as compact form as possible however do not attempt to find a closed form solution.
- P4.** (25 points.) Consider an  $M/M/1$  queue in equilibrium where arrivals correspond to a homogeneous Poisson process with intensity  $\lambda$  and service time is exponentially distributed with parameter  $\mu > \lambda$ ; in particular, the stationary distribution of the number of people in the queue (i.e. in line or undergoing service) is given by the formula

$$\pi(n) = \left(1 - \frac{\lambda}{\mu}\right) \cdot \left(\frac{\lambda}{\mu}\right)^n, \text{ for } n \geq 0.$$

- (a) What's the probability that between time  $t = 5$  and  $t = 7$  nobody leaves the queue? Simplify your answer.
- (b) For a given integer  $n \geq 0$ , what's the probability that there are  $n$  people in the queue when a new person arrives?
- (c) Let  $W$  be the random amount of time that a person waits in line from the moment he arrives to the moment he begins service. Deduce mathematically that

$$P[W = x | W > 0] = (\mu - \lambda)e^{-(\mu - \lambda)x}, \text{ for } x > 0.$$

**Hint:** The density of a Gamma distribution with parameters  $(n, \mu)$  is  $\mu e^{-\mu x} \frac{(\mu x)^{n-1}}{(n-1)!}$  for  $x > 0$ .

**Duration: 50 minutes**

$$\boxed{P1} \quad (a) = P[N_7 - N_5 = 2 \mid N_2 = 5]$$

$$\stackrel{\text{indep. increments}}{=} P[N_7 - N_5 = 2] = \frac{(2\lambda)^2}{2!} e^{-2\lambda} = \boxed{2\lambda^2 e^{-2\lambda}}$$

$$(b) = P[N_7 - N_5 = 2 \mid N_9 = 5]$$

$$\stackrel{\text{conditioning}}{=} \boxed{\binom{5}{2} \left(\frac{2}{9}\right)^2 \left(\frac{7}{9}\right)^3}$$

$\boxed{P2}$

Let  $B \stackrel{\text{def}}{=} \text{time until next bus arrives}$ .

Since  $E(B) = 1/4$ , the memoryless property implies that

$$\boxed{B \sim \text{Exp}(4)}$$

Observe,

$$P[\text{student takes the bus} \mid B=t] = P[\text{HPP with intensity } 6 \cdot \frac{1}{6} \text{ has no arrivals on } [0,t]]$$

$$= \frac{t^0}{0!} e^{-t} = e^{-t}$$

Hence

$$P[\text{student takes the bus}] = \int_0^{\infty} e^{-t} \cdot P[B=t] dt$$

$$= \int_0^{\infty} e^{-t} \cdot 4 e^{-4t} dt = \boxed{\frac{4}{5}}$$

$\boxed{P3} \quad (a)$  The jump-rates of  $X$  are

$$\begin{aligned} q(i, i+1) &= \lambda, \quad 0 \leq i \leq m-1 \\ q(i, i-1) &= i\mu, \quad 1 \leq i \leq m \\ \text{All other jump-rates are } 0 \end{aligned}$$

$$\textcircled{b} = \pi(0) = \frac{\alpha(0)}{\sum_{i=0}^{\infty} \alpha(i)}, \text{ where}$$

$$\alpha(i) = \begin{cases} \lambda, & i=0 \\ \frac{\lambda^i}{i! \prod_{j=1}^i (\mu - \lambda)}, & i \geq 1 \end{cases} = \frac{1}{i!} \left(\frac{\lambda}{\mu}\right)^i, \quad 0 \leq i < \infty$$

Therefore

$$\textcircled{b} = \frac{1}{\sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{\lambda}{\mu}\right)^i} = \left[ \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{\lambda}{\mu}\right)^i \right]^{-1} \quad \#$$

$$\boxed{P4} \quad \textcircled{a} \quad \textcircled{=} \quad P[N_4 - N_3 = 0] = \frac{(2\lambda)^0}{0!} e^{-2\lambda} = \boxed{e^{-2\lambda}}$$

output process is HPP with intensity  $\lambda$

$$\textcircled{b} \quad \textcircled{=} \quad \pi(\infty) = \boxed{\left(1 - \frac{\lambda}{\mu}\right) \cdot \left(\frac{\lambda}{\mu}\right)^{\infty}}$$

equilibrium

$$\textcircled{c} = \frac{P[W=x, W > 0]}{P[W \geq 0]} = \frac{P[W=x]}{1 - P[W=0]} = \frac{P[W=x]}{1 - \pi(0)} = \frac{\mu \cdot P[W=x]}{\lambda}$$

$$= \frac{\lambda}{\lambda} \sum_{n=1}^{\infty} P[W=x \mid \text{there are } n \text{ people in queue}] \cdot P[\text{there are } n \text{ people in queue when person arrives}]$$

$$= \frac{\lambda}{\lambda} \sum_{n=1}^{\infty} \lambda \cdot e^{-\mu x} \frac{(\mu x)^{n-1}}{(n-1)!} \cdot \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^n$$

$$= (\mu - \lambda) e^{-\mu x} \sum_{n=1}^{\infty} \frac{(\mu x)^{n-1}}{(n-1)!} \left(\frac{\lambda}{\mu}\right)^{n-1}$$

$$= (\mu - \lambda) e^{-\mu x} \sum_{n=1}^{\infty} \frac{(\lambda x)^{n-1}}{(n-1)!} = (\mu - \lambda) e^{-\mu x} \cdot e^{\lambda x}$$

$$= \boxed{(\mu - \lambda) e^{-(\mu - \lambda)x}} \quad \#$$