APPM 4/5560 - Laboratory 2 - Fall 2008

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Instructions: This lab is due at the beginning of lecture on Wednesday, November 05, 2008. Students registered for APPM 4560 are encouraged to work in groups of up to three members and submit one lab report with all participant names on it. All participants should be equally responsible and involved in the lab assignment. **Students registered for APPM 5560 must solve the lab on their own.** By submitting a report, all its participants agree to comply with the CU Honor Code Policy.

To receive full credit in this lab you must submit a professional report addressing all the instructions and questions of the lab **in the same order** as they are requested. Be sure to **include all figures or tables and to label each of them** (e.g. Figure 1, Table 2, etc.). Also be sure to **include all algorithms requested**. Algorithms should follow the same **format used in handout #1** and should **relay only on the simulation of uniform random variables** on the open interval $(0, 1)$. Your write-up should **include brief but complete answers** to all the questions listed below **with appropriate references to the labeled figures or tables**. Do not include an introduction nor a conclusion section in your report.

As a recommendation: (a) **before starting the lab read it completely**, and (b) **concentrate first on the theoretical questions and then implement the requested simulations**.

1 Simulating a homogeneous Poisson process (HPP).

Consider the following algorithm to simulate the arrival times of a HPP with intensity $\lambda > 0$ in the interval $[0, t]$.

Algorithm 1

Step 1:	Set $i := 0$ and $T(0) := 0$.
Step 2:	Generate $U \sim Uniform(0, 1)$.
Step 3:	Set $i := i + 1$ and $T(i) := T(i-1) - \ln(U)/\lambda$.
Step 4:	If $T(i) > t$, set $N := i - 1$ and stop. Otherwise, go to Step 2.

Address the following questions in your lab report.

(a.1) What do the random variables $T(1), \ldots, T(N)$ generated by Algorithm 1 represent? Explain.

(a.2) What's the distribution of the random number N? Explain.

(a.3) What does the random quantity $T(N + 1)$ represent? Explain.

(a.4) What's the distribution of the random quantity $T(N + 1) - t$? Explain.

(a.5) Do the random variables $T(N + 1) - T(N)$ and $T(N + 1) - t$ have the same distribution? Explain.

(a.6) Determine the p.d.f. of $T(N+1)$. Include the whole calculation in your report.

(a.7) Implement Algorithm 1 with $\lambda = 1$ and $t = 3$ in any software that has a random number generator to obtain 10K simulations of the random vector $(N, T(N), T(N+1))$. Use the 10K draws to obtain the histograms associated to the quantities N, $T(N+1)-T(N)$, $T(N+1)-t$, and $T(N+1)$.

(a.7.1) Do the generated values of N support your answer to part (a.2)? Comment on any expected and/or unexpected behavior.

(a.7.2) Do the generated values of $T(N+1) - T(N)$ and $T(N+1) - t$ support your answer to part (a.5)? Comment on any expected and/or unexpected behavior.

(a.7.3) Do the generated values of $T(N+1)$ support your answer to part (a.6)? Comment on any expected and/or unexpected behavior.

2 Simulating a non-homogeneous Poisson process (NHPP).

In this second part of the lab you will learn how to simulate a NHPP in a time interval of the form $[0, T]$, where $T > 0$ is a fixed time, using the following result.

Theorem A: *Suppose that $T_1, T_2, \ldots$ correspond to the random arrival times of a HPP with intensity $\lambda > 0$. If $U_1, U_2, \ldots$ are i.i.d. Uniform[0,1] and independent of the arrival times $T_1, T_2, \ldots$ then the random scatter of points $\{(T_i, U_i) : i = 1, 2, \ldots\}$ corresponds to a spatial Poisson process (SPP) in the region $\{(x, y) : x \geq 0$ and $0 \leq y \leq 1\}$ with intensity function $\lambda(x, y) = \lambda$.*

Now consider a NHPP $N(t)$ with a continuous intensity function $\lambda_N(t)$ and let $C > 0$ be a constant such that

$$0 \leq \lambda_N(t) \leq C, \text{ for all } t \in [0, T].$$

To simulate the process $N(t)$, for $t \in [0, T]$, consider the random arrival times $T_1, T_2, \ldots$ of a HPP with intensity C and an i.i.d. sequence of Uniform[0,1] random variables $U_1, U_2, \ldots$ that are independent of the arrival times $T_1, T_2, \ldots$

Address the following questions in your lab report.

(b.1) Consider the process $M(t) := \#\left\{i : T_i \leq t \text{ and } U_i \leq \frac{\lambda_N(T_i)}{C}\right\}$, where it is understood that $M(0) = 0$. Use Theorem A to show almost directly that for all $s, t > 0$,

$$M(s+t) - M(t) \sim \text{Poisson}\left(\int_t^{s+t} \lambda_M(x)dx\right),$$

where $\lambda_M(t) := \min\{C, \lambda_N(t)\}$.

(b.2) Use Theorem A to show almost directly that the process $M(t)$ has independent increments i.e. for all $t_n > \ldots > t_1 > t_0 \geq 0$, $M(t_n) - M(t_{n-1}), \ldots, M(t_1) - M(t_0)$ are independent random variables.

(b.3) Show that for all $t \in [0, T]$, $\lambda_M(t) = \lambda_N(t)$.

From (b.1) and (b.2) it follows that $M(t)$ is a NHPP with intensity function $\lambda_M(t)$. Furthermore, (b.3) states that $M(t)$ and $N(t)$ have the same intensity function in the time interval $[0, T]$. In particular, to simulate $N(t)$, for $t \in [0, T]$, it is enough to simulate $M(t)$, for $t \in [0, T]$. The process $M(t)$ is easy to simulate because we can easily simulate the sequence of Uniform[0,1] random variables $U_1, U_2, \ldots$ and the arrival times $T_1, T_2, \ldots$ of a HPP with intensity C (using Algorithm 1).

In what remains of this lab

$$\lambda(t) := (2t^2 - 10t + 13)/4.$$

(b.4) Use the above insights to design a simple algorithm to simulate a NHPP with intensity function $\lambda(t)$ in the time interval $[0, 4]$.

(b.5) Let W be the random number of arrivals in the time interval $[0, 4]$ of a NHPP with intensity function $\lambda(t)$. What is the theoretical distribution of W? What is the $E(W)$? Explain.

(b.6) Implement the algorithm of part (b.4) to simulate 10K independent draws of W. Does the histogram of the simulated values support your answer in part (b.5)? Is the sample average of the simulated values comparable to the theoretical expected value of W? Comment on any expected and/or unexpected behavior.