APPM 4/5560

Solutions to Review Problems for Exam One, Part II

15. The Markov chain on states S and R has probability transition matrix

$$\mathbf{P} = \begin{array}{c} \\ S \\ R \end{array} \begin{array}{c} \begin{array}{cc} S & R \end{array} \\ \left\| \begin{array}{cc} 0.9 & 0.1 \\ 0.8 & 0.2 \end{array} \right\| \end{array}$$

The answer is π_S, where π_S is the solution to the following system of equations.

$$\begin{array}{ccccc} \pi_S & = & 0.9\pi_S & + & 0.8\pi_R \\ \pi_R & = & 0.1\pi_S & + & 0.2\pi_R \\ \pi_S & + & \pi_R & = & 1 \end{array}$$

The answer is $\pi_S = 8/9$.

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16. The communication classes are $\{0\}$, $\{1, 2\}$, $\{3, 4\}$, and $\{5\}$. Since we did not get one big communication class, the chain is not irreducible.

As for recurrence and transience,

State 0: State 0 is recurrent. From state 0 we will always return to state 0 in one step with probability 1.

States 1 and 2: States 1 and 2 are recurrent. There is no escaping these states!

States 3 and 4: States 3 and 4 are transient. From this class it is possible to get to state 0 after which there is no coming back!

State 5: State 5 is recurrent. From state 5 we will always return to state 5 in one step with probability 1.

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17. (a)

$$\mathbf{P} = \begin{array}{c} \\ R \\ G \\ Y \\ B \end{array} \begin{array}{c} \begin{array}{cccc} R & G & Y & B \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/7 & 1/7 & 4/7 & 1/7 \\ 1/7 & 2/7 & 4/7 & 0 \end{array} \right\| \end{array}$$

(b) Let $T = \min\{n \geq 0 : X_n = R \text{ or } X_n = G\}$. We want to find $P(X_T = R | X_0 = Y)$. For $i = R, G, Y, B$, let $p_i = P(X_T = R | X_0 = i)$. Then we want p_Y.

$$p_Y = \frac{1}{7}(1) + \frac{1}{7}(0) + \frac{4}{7}p_Y + \frac{1}{7}p_B$$

(Note that I have used that $p_R = 1$ and that $p_G = 0$.)
In order to find p_Y, we still need p_B:

$$p_B = \frac{1}{7}(1) + \frac{2}{7}(0) + \frac{4}{7}p_Y.$$

18. Create a new Markov chain with 1 as an absorbing state. It will have transition matrix

$$\mathbf{Q} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 \end{array} \right\| \end{array}$$

The new chain acts exactly like the old chain until state 1 is hit. At this point the new chain "sticks" in state 1.

Now

$$P(X_5 = 2 | X_4 \neq 1, X_3 \neq 1, X_2 \neq 1, X_1 \neq 1, X_0 = 0)$$

$$= \frac{P(X_5 = 2, X_4 \neq 1, X_3 \neq 1, X_2 \neq 1, X_1 \neq 1 | X_0 = 0)}{P(X_4 \neq 1, X_3 \neq 1, X_2 \neq 1, X_1 \neq 1 | X_0 = 0)}$$

The numerator is the probability, starting from state 0, that the original chain goes to state 2 in 5 time steps without ever going through state 1. This is the same as the probability that the "Q chain" goes from state 0 to state 2 in 5 time steps. This is $q_{02}^{(5)}$.

The denominator is the probability that, starting from state 0, the original chain does not hit state 1 in 4 time steps. This is 1 minus the probability that the original chain, starting in state 0, hits state 1 at least 1 time in 4 time steps. If the original chain hits state 1 in 4 time steps then the "Q chain" will be stuck in state 1 at 4 time steps. Therefore, the denominator is $1 - q_{01}^{(4)}$.

In summary,

$$P(X_5 = 2 | X_4 \neq 1, X_3 \neq 1, X_2 \neq 1, X_1 \neq 1, X_0 = 0) = \frac{q_{02}^{(5)}}{1 - q_{01}^{(4)}}.$$