APPM 4/5560

Solutions to Review Problems for Exam Two

1. (a) Let $T = \min\{n \geq 0 : X_n = B \text{ or } X_n = C\}$. Let $u_i = \mathsf{E}[T|X_0 = i]$. We want to find u_A. Starting out with at least one purchase,

$$u_A = 1 + 0.6u_A + 0.2(0) + 0.2(0)$$

(Those zeros are because we have switched brands so we stop counting.)
So, we have that $u_A = 2.5$ purchases.

(b) Let w_{ij} be the mean time. starting from state i, spent in state j, before hitting state C. We want to find w_{AB}.

$$w_{AB} = 0.6w_{AB} + 0.2w_{BB} + 0.2(0)$$

Note that we did not start out using "1+" since we are not counting purchases (time steps) but rather purchases of brand B. We may go from brand A to brand C without ever hitting B at all! (Even if the matrix was such that we are forced to go through B on the way from A to C, we would not count hits on B until we actually hit B as in the next equation.)
Now

$$w_{BB} = 1 + 0.1w_{AB} + 0.7w_{BB} + 0.2(0)$$

Solving this system of 2 equations, gives us

$$\boxed{w_{AB} = 2} \qquad \text{and} \qquad w_{BB} = 4.$$

(c) Let w_{ij} be the mean time. starting from state i, spent in state j, before hitting state A again. We want to find w_{AB}.
Even though we are starting at A, we want to stop counting if we make a move to A, so, we get

$$w_{AB} = 0.6(0) + 0.2w_{BB} + 0.2w_{CB}$$

and

$$w_{BB} = 1 + 0.1(0) + 0.7w_{BB} + 0.2w_{CB}$$

and

$$w_{CB} = 0.1(0) + 0.1w_{BB} + 0.8w_{CB}.$$

Solving this system of 3 equations gives us

$$w_{AB} = 1.5, \qquad w_{BB} = 5, \qquad \text{and} \quad w_{CB} = 2.5.$$

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2. Let $X \sim exp(rate = \lambda)$. The lack of memory property of the exponential says that, for any $s, t > 0$,

$$P(X > s + t|X > s) = P(X > t).$$

In words, if X represents the time until something happens, if we have already waited at least s units of time, the probability that we will have to wait at least t units more is the probability we have to wait at least t units in the first place. It is as if the fact that we already waited more than s units of time is irrelevant!

Proof:

$$P(X > s+t|X > s) \quad = \quad \frac{P(X>s+t, X>s)}{P(X>s)} = \frac{P(X>s+t)}{P(X>s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t).$$

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3. (a)
$$\begin{aligned} P(N(2)=3, N(6)=3) &= P(N(2)=3, N(6)-N(2)=0) \\ &\overset{indep}{=} P(N(2)=3)\cdot P(N(6)-N(2)=0) \\ &= \frac{e^{-6}6^3}{3!}\cdot\frac{e^{-12}12^0}{0!} \end{aligned}$$

(b)
$$\mathsf{E}[N(3)-N(2)|N(2)\geq 1] \overset{indep}{=} \mathsf{E}[N(3)-N(2)] = (3)(1) = 3$$

(c)
$$P(N(2)=3|S_2=1.8) = P(N(2)-N(1.8)=1) = \frac{e^{-3(0.2)}(3(0.2))^1}{1!}$$

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4. (a) Forget all about the cars that won't stop. The cars that will pick up the hitchhiker follow a Poisson process (thinned from the original) with rate $\lambda_1 = (200)(0.05) = 10$ cars per hour. The amount of time the hitchhiker has to wait is the expected value of one of the exponential, rate 10, interarrival times. This is one-tenth of an hour or six minutes.

(b) Let T be the time until the hitchhiker gets picked up. We know that

$$T \sim \exp(rate = 1/10)$$

Let $N_2(t)$ be the number of cars that didn't stop by time t. This is a thinned Poisson process with rate $\lambda_2 = (200)(0.95) = 190$.
We want

$$\mathsf{E}[N_2(T)].$$

Well

$$\begin{aligned} \mathsf{E}[N_2(T)] &= \textstyle\int_0^\infty \mathsf{E}[N_2(T)|T=t]\cdot f_T(t)\,dt \\ &= \textstyle\int_0^\infty \mathsf{E}[N_2(t)|T=t]\cdot 10e^{-10t}\,dt \\ &= \textstyle\int_0^\infty \mathsf{E}[N_2(t)]\cdot 10e^{-10t}\,dt \end{aligned}$$

That last equality follows from the fact that thinning a Poisson process into two Poisson processes produces independent processes. $N_2(t)$ is a quantity from the second process and T is a quantity from the first process.

Continuing...

$$
\begin{aligned}
\mathsf{E}[N_2(T)] &= \mathsf{E}[N_2(t)] \cdot 10e^{-10t}\,dt \\
&= (190t) \cdot 10e^{-10t}\,dt \\
&= 1900te^{-10t}\,dt \\
&= 19
\end{aligned}
$$

Note: This is what we expect since the expected time for a pickup is 1/10 of an hour and the expected number of cars passing in 1/10 of an hour is $190(1/10) = 19$. Warning: One could not use this more naive (yet intuitive!) approach if the two processes were dependent.

(c) This is very similar to part (b). It is not worded ver well though. Are the partol cars included in ones that will pick up or ignore the hitchiker? This would have to be worded better on an actual exam. For this solution, I'm going to have three distinct (non-overlapping) groups:

- 1: cars that will pick up the hitchiker
- 2: cars that will pass the hitchiker
- 3: patrol cars that will bother the hitchiker

My interpretation is that
$N_1(t)$ is a Poisson process with rate $(200)(0.05) = 10$ cars per hour.
$N_3(t)$ is a Poisson process with rate $(200)(1/100)(0.7) = 1.4$ cars per hour.
$N_2(t)$ is a Poisson process with the remaining rate $200 - 10 - 1.4 = 188.6$ cars per hour.
Letting T_3 be the first time the hitchiker gets bothered by police, and T_1 be the first time the hitchiker gets picked up, we want

$$P(T_1 < T_3).$$

Now this is like many problems we have had (but easier– exponentials in place of gammas!).
We have

$$T_1 \sim \exp(rate = 10), \qquad T_3 \sim \exp(rate = 1.4)$$

where T_1 and T_2 are independent.

$$
\begin{aligned}
P(T_1 < T_3) &= \textstyle\int_0^\infty P(T_1 < t)\, 1.4e^{-1.4t}\,dt \\
&= \textstyle\int_0^\infty \int_0^t 10e^{-10u}\,du\, 1.4e^{-1.4t}\,dt \\
&= \cdots \approx 0.87719298
\end{aligned}
$$

Note: We didn't have to do an integral at all. Ignoring the type 2 cars. The pickup cars and patrol cars make one Poisson process with rate $10 + 1.4 = 11.4$. The probability that the next arrival in this process is a pickup car is simply

$$\frac{10}{11.4} \approx 0.87719298.$$

In fact, in general, for $X_1 \exp(rate = \lambda_1)$ and independent $X_2 \exp(rate = \lambda_2)$, we always have that

$$P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

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5. The maximum of two numbers is below a value y if and only if each number is below the value y. Therefore, we get the second inequality in

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(\max\{X_1, X_2\} \leq y) \\ &= P(X_1 \leq y, X_2 \leq y) \\ &\overset{indep}{=} P(X_1 \leq y) \cdot P(X_2 \leq y) \\ &\overset{ident}{=} [P(X_1 \leq y)]^2 \\ &= [1 - e^{-\lambda y}]^2 \end{aligned}$$

So, the pdf is

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}[1 - e^{-\lambda y}]^2 = 2\lambda e^{-\lambda y}\,[1 - e^{-\lambda y}]$$

for $y > 0$.

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6. (a) The arrival time is uniformaly distributed between 0 and 60, so, the desired probability is

$$\frac{20}{60} = \frac{1}{3}.$$

(b) Let $X =$ the number that arrive during the first hour. Then $X \sim bin(2, 1/3)$.

$$P(X = 1) = \binom{2}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^1$$

(c)

$$\begin{aligned} P(\text{ at least one }) &= P(X \geq 1) \\ &= 1 - P(X = 0) = 1 - \binom{2}{0}\left(\tfrac{1}{3}\right)^0\left(\tfrac{2}{3}\right)^2 \end{aligned}$$

7.

$$P(X = x | X + Y = n) \;=\; \frac{P(X=x, X+Y=n)}{P(X+Y=n)} = \frac{X=x, Y=n-x}{P(X+Y=n)}$$

Since X is a Poisson random variable, this probability will be zero if x is less than 0. Since Y is a Poisson random variable, this probability will also be zero if $n - x$ gets negative. So, continuing under the assumption that $x = 0, 1, 2, \ldots, n$, and using the fact that the sum of independent Poisson random variables is again Poisson with rate parameter equal to the sum of the individual rate paremters (this can be/was shown using moment generating functions), we have

$$\begin{aligned}
P(X = x | X + Y = n) &= \tfrac{X=x, Y=n-x}{P(X+Y=n)} \\
&\stackrel{indep}{=} \tfrac{P(X=x) \cdot P(Y=n-x)}{P(X+Y=n)} \\
&= \frac{\frac{e^{-\lambda_1}\lambda_1^x}{x!} \cdot \frac{e^{-\lambda_2}\lambda_2^{n-x}}{(n-x)!}}{\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}} \\
&= \tfrac{n!}{x!(n-x)!} \left(\tfrac{\lambda_1}{\lambda_1+\lambda_2}\right)^x \left(\tfrac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-x} \\
&= \begin{pmatrix} n \\ x \end{pmatrix} \left(\tfrac{\lambda_1}{\lambda_1+\lambda_2}\right)^x \left(1 - \tfrac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-x}
\end{aligned}$$

for $x = 0, 1, 2, \ldots, n$. The probability is 0 otherwise.

So, $X|X + Y = n \sim bin(n, \lambda_1/(\lambda_1 + \lambda_2))$.

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8. In order to take advantage of "independent increments", let us rewrite this as

$$\begin{aligned}
\mathsf{E}[N(t)N(s+t)] &= \mathsf{E}[N(t) \cdot (N(s+t) - N(t)) + N^2(t)] \\
&= \mathsf{E}[N(t) \cdot (N(s+t) - N(t))] + \mathsf{E}[N^2(t)]
\end{aligned}$$

The first expectation can be computed as

$$\mathsf{E}[N(t) \cdot (N(s+t) - N(t))] = \mathsf{E}[N(t)] \cdot \mathsf{E}[(N(s+t) - N(t))] = \lambda t \cdot \lambda s = \lambda^2 st.$$

The second one is

$$\mathsf{E}[N^2(t)] = Var[N(t)] + (\mathsf{E}[N(t)])^2 = \lambda t + (\lambda t)^2.$$

So,

$$\mathsf{E}[N(t)N(s+t)] = \lambda^2 st + \lambda t + (\lambda t)^2.$$

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9. (a) If 1 minor defect is found in the first 10 feet, it's location is uniformly distributed over those 10 feet, so the probability it is in the first 2 feet is $2/10 = 1/5$.

(b)

$$
\begin{aligned}
P(N_1(10)=1|N(10)=1) &= \tfrac{P(N_1(10)=1,N(10)=1)}{P(N(10)=1)} \\
&= \tfrac{P(N_1(10)=1,N_2(10)=0)}{P(N(10)=1)} \\
&= \tfrac{P(N_1(10)=1)\cdot P(N_2(10)=0)}{P(N(10)=1)} \\
&= \tfrac{\frac{e^{-10\lambda_1}(10\lambda_1)}{1!}\cdot\frac{e^{-10\lambda_2}(10\lambda_2)^0}{0!}}{\frac{e^{-10(\lambda_1+\lambda_2)}[10(\lambda_1+\lambda_2)]^1}{1!}} \\
&= \tfrac{\lambda_1}{\lambda_1+\lambda_2}.
\end{aligned}
$$

Note: You did not need to do this "from scratch". Since the big overall defect process is a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$, it can be thinned down to a minor defect process by assigning defects as "minor" with probability p. This means that the minor defect process has rate λp. Since we already know that the minor defect process has rate λ_1, we then know that

$$p = \frac{\lambda_1}{\lambda} = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

(c) Um, well, okay... zero! There are no minor defect between two successive minor defects! But, let's suppose that the question was:

> "What is the expected number of minor defects between two successive major defects?"

The time between two successive major defects is an exponential random variable with rate λ_2. So, we want the expected number of minor defects in an interval of length $T \sim exp(rate = \lambda_2)$. Since the Poisson process is stationary, we can just consider the expected number of minor defects in the first T units of time. That, is, we want

$$
\begin{aligned}
\mathsf{E}[N_1(T)] &= \textstyle\int_0^\infty \mathsf{E}[N_1(T)|T=t]\cdot f_T(t)\,dt \\
&= \textstyle\int_0^\infty \mathsf{E}[N_1(t)|T=t]\cdot f_T(t)\,dt \\
&= \textstyle\int_0^\infty \mathsf{E}[N_1(t)]\cdot f_T(t)\,dt
\end{aligned}
$$

since $\{N_1(t)\}$ and $\{N_2(t)\}$ are independent Poisson processes and since T is an interarrival time for the $\{N_2(t)\}$ process.

Continuing,

$$
\begin{aligned}
\mathsf{E}[N_1(T)] &= \textstyle\int_0^\infty \mathsf{E}[N_1(t)]\cdot f_T(t)\,dt \\
&= \textstyle\int_0^\infty \lambda_1 t\cdot f_T(t)\,dt = \lambda_1\int_0^\infty t\cdot f_T(t)\,dt \\
&= \lambda_1\mathsf{E}[T] = \lambda_1/\lambda_2.
\end{aligned}
$$

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10. (a)

$$P(N(1) = 5)|N(2) = 12) = \left(\begin{array}{c} 12 \\ 5 \end{array} \right) \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right)^7$$

Alternatively, you can write

$$\begin{aligned} P(N(1) = 5)|N(2) = 12) &= \tfrac{P(N(1)=5, N(2)=12)}{P(N(2)=12)} \\ &= \tfrac{P(N(1)=5, N(2)-N(1)=7)}{N(2)=12}) = \cdots \end{aligned}$$

(b) Since, by time 1, we have the first 3 arrivals, the total expected time to wait for 5 arrivals is 1 plus the expected time to wait for 2 arrivals.

$$\mathsf{E}[S_5|N(1) = 3] = 1 + \mathsf{E}[S_2] = 1 + 2/\lambda$$

Since $S_2 \sim \Gamma(2, \lambda)$.

(c) Let $T_1, T_1, ldots$ be the interarrival times. (So, $S_n = T_1 + \cdots + T_n$.) First, note that $S_1 = T_1$. So

$$\begin{aligned} \mathsf{E}[S_5|S_1 > t] &= \mathsf{E}[S_5|T_1 > t] \\ &= \mathsf{E}[T_1 + T_2 + T_3 + T_4 + T_5|T_1 > t] \\ &= \mathsf{E}[T_1|T_1 > t] + \mathsf{E}[T_2|T_1 > t] + \cdots + \mathsf{E}[T_5|T_1 > t] \\ &= \mathsf{E}[T_1|T_1 > t] + \mathsf{E}[T_2] + \cdots + \mathsf{E}[T_5] \end{aligned}$$

since the interarrival times are independent. By the lack of memory property of the exponential, $T_1|T_1 > t \sim exp(rate = \lambda)$. So $\mathsf{E}[T_1|T_1 > t] = \mathsf{E}[T_1]$. So

$$\begin{aligned} \mathsf{E}[S_5|S_1 > t] &= \mathsf{E}[T_1] + \mathsf{E}[T_2] + \cdots + \mathsf{E}[T_5] \\ &= \tfrac{1}{\lambda} + \cdots + \tfrac{1}{\lambda} = \tfrac{5}{\lambda}. \end{aligned}$$

(d) $\mathsf{E}[N(t)|S_1 > t] = \mathsf{E}[N(t)|T_1 > t] = 0$ since the first event happens at a time greater than t therefore no events have happened by time t!

(e) Since $\mathsf{E}[N(t)|S_1 < t] = \mathsf{E}[N(t)|T_1 < t]$, we know at least one event has happened before time t. One way to do this problem is to condition on the time of that first event. (Note that you would have to use the conditional density for T_1 given that $T_1 < t$.) Alternatively,

$$\begin{aligned} \mathsf{E}[N(t)] &= \mathsf{E}[N(t)|T_1 < t] \cdot P(T_1 < t) + \mathsf{E}[N(t)|T_1 > t] \cdot P(T_1 > t) \\ &= \mathsf{E}[N(t)|T_1 < t] \cdot P(T_1 < t) + 0 \cdot P(T_1 > t) \\ &= \mathsf{E}[N(t)|T_1 < t] \cdot P(T_1 < t) \end{aligned}$$

which implies that

$$\mathsf{E}[N(t)|T_1 < t] = \frac{\mathsf{E}[N(t)]}{P(T_1 < t)} = \frac{\lambda t}{1 - e^{-\lambda t}}.$$

11. X has the same distribution as the time of the nth arrival of a Poisson process with rate λ. Therefore,

$$P(X \leq x) \quad = \quad P(S_n \leq x).$$

Well,

$$P(S_n > t) = P(N(t) < n) = \sum_{k=0}^{n-1} P(N(t) = k) = \sum_{k=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!},$$

so

$$\begin{aligned} P(X \leq x) &= P(S_n \leq x) = 1 - P(S_n > x) \\ &= 1 - \textstyle\sum_{k=0}^{n-1} \frac{e^{-\lambda x}(\lambda x)^k}{k!}. \end{aligned}$$