APPM 4/5560 Markov Chains

Fall 2009, Some Review Problems for the Final

1. Consider a two-server ququeing process where customers arrive according to a Poisson process with rate λ, and go to either one of two servers if they are available. If both servers are busy, the customers form a single queue where the person at the end of the line will go to the next available server.

   Suppose that each sever takes an exponential amount of time with rate μ to serve a customer.

   If we model this as a birth-and-death process, what are the birth and death rates?

2. Suppose that in an M/M/1 queue, the customer arrival rate is 3 per minute. Find the service rate so that 95% of the time the queue will contain less than 10 customers.

3. Sal the barber has a shop that can hold up to 3 customers, including the one being served. Excess customers get turned away. If customer interarrival times are exponential with rate λ, and Sal, as the only worker in the shop can serve people at a rate that is exponential also with rate λ, find the expected number of customers in the shop.

4. For an M/M/1 queue in equilibrium with interarrival rate λ and service rate μ, find the expected time between two consecutive times the queue is empty.

5. Consider a Poisson process $\{N(t)\}$ with rate λ with rateλ for some $\lambda > 0$.

   Does this process have a stationary distribution? If so, find it. If not, explain why it does not.

6. Suppose that d particles are distributed into two cells. A particle in cell 1 remains in that cell for a random length of time that is exponentially distributed with rate λ before moving to cell 2. A particle in cell 2 remains in that cell for a random length of time that is exponentially distributed with rate μ before moving to cell 1. The particles act independently of each other. Let $X(t)$ denote the number of paticles in cell 2 at time $t \geq 0$. The $X(t)$ is a birth-and-death process on $\{0, 1, 2, \ldots, d\}$.

   (a) Find the birth-and-death rates.
   (b) Find the stationary distribution for $X(t)$.
   (c) When the system is in equlibrium, find $\mathsf{E}[X(t)]$.

7. For the M/M/1 queue, find the mean queue length in equilibrium

   (a) "from scratch"
   (b) using the formula I derived for the M/G/1 queue that is posted on Monday Dec 7th

8. A very important "balance flow" equation in queueing theory is

   $$L = \lambda W$$

   where L is the mean queue length in equilibrium, λ is the customer arrival rate, and W is the average time spent by a customer in the system.

   Verify this balance equation for the M/M/1 queue by computing W "from scratch".

9. Determine the mean waiting time for a customer in an M/M/2 system when $\lambda = 2$ and $\mu = 1.2$. Compare this with the mean waiting time for a customer in an M/M/1 system with $\lambda = 1$ and $\mu = 1.2$. Why is there a difference when the arrival rate per server is the same in both cases?

10. Consider the M/G/1 queue, where the service times have a $\Gamma(2, \nu)$ distribution. Assume the system is in equilibrium.

    (a) What is the probability that an incoming customer has to wait for service?

    (b) What is the expected queue length in equilibrium?

11. Consider a simple M/M/1 queue with arrival rate λ and service rate μ. Find the distribution of the number of customers who arrive while a given customer is being served. Assume the queue is in equilibrium.

12. Consider an M/M/s queue in equilibrium with the normal arrival and service rate parameters λ and μ. Let L be the mean number of customers in the system and let L_0 be the mean number of customers in the system waiting for, but not undergoing, service.

    One can show (but you don't have to) that

$$L_0 = \frac{\pi_0}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{(\lambda/s\mu)}{(1-\lambda/s\mu)^2}$$

    Relate L and L_0. That is, fill in the question mark in

$$L = L_0 + ?$$