APPM 4/5560

Solutions to Review Problems for the Final

1.

$$\lambda_i = \lambda, \qquad i = 0, 1, 2, \ldots$$

$$\mu_1 = \mu$$

$$\mu_i = 2\mu, \qquad i = 2, 3, \ldots$$

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2. We want

$$0.95 = \sum_{n=0}^{9} \pi_n$$

where

$$\pi_n = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n, \qquad n = 0, 1, 2, \ldots$$

and $\lambda = 3$.

$$\sum_{n=0}^{9} \pi_n = \left(1 - \frac{\lambda}{\mu}\right)\sum_{n=0}^{9}\left(\frac{\lambda}{\mu}\right)^n = 1 - \left(\frac{\lambda}{\mu}\right)^{10}$$

So, we need to solve

$$1 - \left(\frac{3}{\mu}\right)^{10} = 0.95$$

for μ. The answer is $\mu = 3/(0.05)^{1/10} \approx 4.048$.

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3. $\lambda_0 = \lambda_1 = \lambda_2 = \lambda$, $\lambda_3 = 0$, $\lambda_4 = 0, \ldots$, etc...

$\mu_1 = \mu_2 = \mu_3 = \lambda$

So the stationary distribution which gives the probability of there being n customers in the shop is, for $n = 0, 1, 2, 3$

$$\pi_n = \frac{\lambda_1 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}\pi_0 = \left(\frac{\lambda}{\lambda}\right)^n \pi_0 = \pi_0$$

So,

$$1 = \pi_0 + \pi_1 + \pi_2 + \pi_3 = \pi_0 + \pi_0 + \pi_0 + \pi_0 = 4\pi_0$$

$$\Rightarrow \pi_0 = \frac{1}{4}$$

and hence $\pi_0 = \pi_1 = \pi_2 = \pi_3 = 1/4$.

So, the expected number of customers in the shop is

$$\sum_{n=0}^{3} n \cdot P(n \text{ customers in the shop}) = \sum_{n=0}^{3} n\pi_n = \frac{1}{4}\sum_{n=0}^{3} n = \frac{1}{4}[0+1+2+3]$$

$$= \frac{3}{2}$$

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4. This was intended to be a very quick problem where you let $T_0 = \min\{n \geq 1 : X_n = 0\}$ and then use the facts that

$$\mathsf{E}_0[T_0] = \frac{1}{\pi_0} = \frac{1}{(1-\lambda/\mu)} = \frac{\mu}{\mu - \lambda}.$$

However, we never got to showing or talking about the first equality above. Given what we have covered, you could do this problem in a long way similar to our recursive computation of $\mathsf{E}[T_i]$ where T_i was defined as the time, starting from i that it takes to first hit state $i+1$. Save it for the winter break!

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5. No, $\{N(t)\}$ does not have a stationary distribution. It is an increasing process, so there is no way that the distribution of its values at time time t_1 could be the same as the distribution of its values at some later time t_2. To express this a bit more formally, one could write down the generator matrix Q and then it is easy to see that, for $\lambda > 0$ the only solution to $\vec{\pi}Q = \vec{0}$ is $\vec{\pi} = \vec{0}$ which is not a distrbution. (It doesn't sum to 1.)

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6. (a) Birth rates: $\lambda_0 = d\lambda$, $\lambda_1 = (d-1)\lambda$, etc... In general,

$$\lambda_i = \begin{cases} (d-i)\lambda & , \quad i = 0, 1, \ldots, d \\ 0 & , \quad \text{otherwise} \end{cases}$$

Death Rates: $\mu_0 = 1$, $\mu_1 = \mu$, $\mu_2 = 2\mu$, etc... In general,

$$\mu_i = \begin{cases} i\mu & , \quad i = 0, 1, \ldots, d \\ 0 & , \quad \text{otherwise} \end{cases}$$

(b) $\pi_n = 0$ for $n > d$. For $n = 0, 1, 2, \ldots, d$,

$$\begin{aligned}
\pi_n &= \tfrac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\,\pi_0 \\
&= \tfrac{(d\lambda)((d-1)\lambda)\cdots((d-(n-1))\lambda)}{(\mu)(2\mu)\cdots(n\mu)}\,\pi_0 \\
&= \left(\tfrac{\lambda}{\mu}\right)^n \tfrac{d!/(d-n)!}{n!}\,\pi_0 \\
&= \left(\tfrac{\lambda}{\mu}\right)^n \tfrac{d!}{n!(d-n)!}\,\pi_0 \\
&= \binom{d}{n}\left(\tfrac{\lambda}{\mu}\right)^n \pi_0
\end{aligned}$$

To find π_0:

$$1 = \pi_0 \sum_{n=0}^{d} \binom{d}{n}\left(\frac{\lambda}{\mu}\right)^n = \pi_0 \sum_{n=0}^{d} \binom{d}{n}\left(\frac{\lambda}{\mu}\right)^n (1)^{d-n} = \pi_0 \left(1 + \frac{\lambda}{\mu}\right)^d$$

where the last equality is by the binomial theorem. Therefore

$$\pi_0 = \left[\left(1+\frac{\lambda}{\mu}\right)^d\right]^{-1}$$

and, for $n = 1, 2, \ldots, d$,

$$\pi_n = \binom{d}{n}\left(\frac{\lambda}{\mu}\right)^n \pi_0.$$

(c)

$$\begin{aligned}
\mathsf{E}[X(t)] &= \textstyle\sum_{n=0}^{d} n\,\pi_n \\
&= \pi_0 \textstyle\sum_{n=0}^{d} n \binom{d}{n} \left(\frac{\lambda}{\mu}\right)^n \\
&= \pi_0 \textstyle\sum_{n=1}^{d} n \binom{d}{n} \left(\frac{\lambda}{\mu}\right)^n \\
&= \pi_0\,(\lambda/\mu) \textstyle\sum_{n=1}^{d} n \binom{d}{n} \left(\frac{\lambda}{\mu}\right)^{n-1} \\
&= \pi_0\,(\lambda/\mu) \frac{d}{dr} \textstyle\sum_{n=1}^{d} \binom{d}{n} r^n
\end{aligned}$$

where $r = \lambda/\mu$.

Now

$$\sum_{n=1}^{d} \binom{d}{n} r^n = \sum_{n=0}^{d} \binom{d}{n} r^n - 1 = (r+1)^d - 1.$$

So,

$$\frac{d}{dr}\sum_{n=1}^{d} \binom{d}{n} r^n = \frac{d}{dr}\left[(r+1)^d - 1\right] = d(r+1)^{d-1}.$$

Therefore

$$\begin{aligned}
\mathsf{E}[X(t)] &= \pi_0(\lambda/\mu)\,d\left(\tfrac{\lambda}{\mu}+1\right)^{d-1} \\
&= \left[\left(1+\tfrac{\lambda}{\mu}\right)^d\right]^{-1} (\lambda/\mu)\,d\left(\tfrac{\lambda}{\mu}+1\right)^{d-1} \\
&= \tfrac{d\cdot\lambda/\mu}{1+\lambda/\mu}
\end{aligned}$$