APPM 4/5560 Markov Chains

Solutions to Review Problems for the Final-Part II

7. (a) Let $X =$ the number of people in the queue in equilibrium. Then

$$P(X=n) = \pi_n = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n, \qquad n = 0,1,2,\ldots$$

So,

$$\begin{aligned}
\mathsf{E}[X] &= \textstyle\sum_{n=0}^{\infty} n \cdot P(X=n) \\
&= \left(1-\tfrac{\lambda}{\mu}\right) \textstyle\sum_{n=0}^{\infty} n \left(\tfrac{\lambda}{\mu}\right)^n \\
&= \left(1-\tfrac{\lambda}{\mu}\right) \textstyle\sum_{n=1}^{\infty} n \left(\tfrac{\lambda}{\mu}\right)^n \\
&= \left(1-\tfrac{\lambda}{\mu}\right) \tfrac{\lambda}{\mu} \textstyle\sum_{n=1}^{\infty} n \left(\tfrac{\lambda}{\mu}\right)^{n-1} \\
&= \left(1-\tfrac{\lambda}{\mu}\right) \tfrac{\lambda}{\mu} \textstyle\sum_{n=1}^{\infty} \tfrac{d}{dq} q^n
\end{aligned}$$

where $q = \lambda/\mu$.
So,

$$\begin{aligned}
\mathsf{E}[X] &= \left(1-\tfrac{\lambda}{\mu}\right) \tfrac{\lambda}{\mu} \tfrac{d}{dq} \textstyle\sum_{n=1}^{\infty} q^n \\
&= \left(1-\tfrac{\lambda}{\mu}\right) \tfrac{\lambda}{\mu} \tfrac{d}{dq} \tfrac{q}{1-q} \\
&= \left(1-\tfrac{\lambda}{\mu}\right) \tfrac{\lambda}{\mu} \tfrac{1}{(1-q)^2} \\
&= \left(1-\tfrac{\lambda}{\mu}\right) \tfrac{\lambda}{\mu} \tfrac{1}{(1-\lambda/\mu)^2} \\
&= \tfrac{\lambda}{\mu-\lambda}
\end{aligned}$$

(b) For the M/G/1 queue, the mean queue length in equlibrium to be

$$L = \frac{2\frac{\lambda}{\mu} + \lambda^2\sigma^2 - \frac{\lambda^2}{\mu^2}}{2\left(1-\frac{\lambda}{\mu}\right)}$$

where σ^2 is the variance of the service time distribution.
In the M/M/1 queue, service times are exponential with rate μ. Hence, they have mean $1/\mu$ and variance $1/\mu^2$. So, L becomes

$$L = \frac{2\frac{\lambda}{\mu} + \lambda^2\frac{1}{\mu^2} - \frac{\lambda^2}{\mu^2}}{2\left(1-\frac{\lambda}{\mu}\right)} = \frac{2\frac{\lambda}{\mu}}{2\left(1-\frac{\lambda}{\mu}\right)} = \frac{\frac{\lambda}{\mu}}{\left(1-\frac{\lambda}{\mu}\right)} = \frac{\lambda}{\mu-\lambda}$$

Yeah!

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8. Assume the M/M/1 queue is in equlibrium. Let W be the average time a customer spends in the system. Let N be the number of customers in the sytem when this customer arrives. Then,

$$\begin{aligned} \mathsf{E}[W] &= \textstyle\sum_{n=0}^{\infty} \mathsf{E}[W|N=n] \cdot P(N=n) \\ \mathsf{E}[W] &= \textstyle\sum_{n=0}^{\infty} \mathsf{E}[W|N=n] \cdot \pi_n \end{aligned}$$

If $N = 0$, the customer will spend only his own service time in the system. As this is exponential with rate μ,

$$\mathsf{E}[W|N=0] = 1/\mu.$$

If $N = 1$, the customer will wait through 1 exponential rate μ service time (lack of memory of the exponential allows us to ignore how long that first customer was there before the customer we are following arrives) and then his own exponential rate μ service time. The sum of these two service times is $\Gamma(2,\mu)$ which has mean $2/\mu$.

In general, if $N = n$, the customer will wait through $n+1$ service times. i.e., he will wait for a $\Gamma(n+1,\mu)$ amount of time which has mean $(n+1)/\mu$.

So,

$$\begin{aligned} \mathsf{E}[W] &= \textstyle\sum_{n=0}^{\infty} \mathsf{E}[W|N=n] \cdot \pi_n \\ &= \textstyle\sum_{n=0}^{\infty} \frac{n+1}{\mu} \cdot \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \\ &= \frac{1}{\mu}\left(1-\frac{\lambda}{\mu}\right) \textstyle\sum_{n=0}^{\infty} \frac{d}{dq} q^{n+1} \end{aligned}$$

where $q = \lambda/\mu$.

Continuing...

$$\begin{aligned} \mathsf{E}[W] &= \frac{1}{\mu}\left(1-\frac{\lambda}{\mu}\right) \frac{d}{dq}\frac{q}{1-q} \\ &= \frac{1}{\mu}\left(1-\frac{\lambda}{\mu}\right) \frac{1}{(1-q)^2} \\ &= \frac{1}{\mu}\left(1-\frac{\lambda}{\mu}\right) \frac{1}{(1-\lambda/\mu)^2} \\ &= \frac{1}{\mu-\lambda}. \end{aligned}$$

Now, since $L = \frac{\lambda}{\mu-\lambda}$ (see problem 7), we can easily see that the balance equation

$$L = \lambda W$$

holds!

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9. Let W be a waiting time of a customer arriving to the M/M/1 queue in equilibrium. Then $\mathsf{E}[W|N=0]=0$, $\mathsf{E}[W|N=1]=1/\mu$, $\ldots$ $\mathsf{E}[W|N=n]=n/\mu$.

So,

$$\begin{aligned}
\mathsf{E}[W] &= \textstyle\sum_{n=0}^{\infty} \mathsf{E}[W|N=n]\cdot\pi_n \\
&= \textstyle\sum_{n=0}^{\infty} \frac{n}{\mu}\cdot\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \\
&= \textstyle\sum_{n=1}^{\infty} \frac{n}{\mu}\cdot\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \\
&= \textstyle\frac{1}{\mu}\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)\sum_{n=1}^{\infty} nq^{n-1} \\
&= \textstyle\frac{1}{\mu}\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)\sum_{n=1}^{\infty} \frac{d}{dq}q^{n} \\
&= \textstyle\frac{1}{\mu}\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right) \\
&= \textstyle\frac{1}{\mu}\frac{\lambda}{\mu}\frac{1}{1-\frac{\lambda}{\mu}} = \frac{\lambda}{\mu(\mu-\lambda)}
\end{aligned}$$

where $q=\lambda/\mu$.

For the M/M/2 system,

$$\mathsf{E}[W|N=0]=0$$
$$\mathsf{E}[W|N=1]=0$$
$$\mathsf{E}[W|N=2]=\frac{1}{2\mu}$$
$$\mathsf{E}[W|N=3]=\frac{1}{2\mu}+\frac{1}{2\mu}=\frac{2}{2\mu}$$

This last one used the lack of memory of the exponential. In this scenario, there are customers at both of the two servers and then another customer in line in front of our customer. We will have an expected wait of $1/2\mu$ for a customer to leave the system and which time the customer ahead of us in the line steps up for service. The other customer (one of the two originally being served), still has an exponential amount of time to go, so we will have another expected wait of $1/2\mu$ until another customer leaves and we can step up to the server.

Continuing in this manner,

$$\mathsf{E}[W|N=4]=\frac{3}{2\mu}$$
$$\vdots$$
$$\mathsf{E}[W|N=n]=\frac{n-1}{2\mu}$$

So,

$$\begin{aligned}
\mathsf{E}[W] &= \textstyle\sum_{n=0}^{\infty} \mathsf{E}[W|N=n]\cdot\pi_n \\
\mathsf{E}[W] &= \textstyle\sum_{n=2}^{\infty} \mathsf{E}[W|N=n]\cdot\pi_n
\end{aligned}$$

for the M/M/2 queue,

$$\pi_n = \frac{1}{2^{n-1}}\left(\frac{\lambda}{\mu}\right)^n \pi_0, \qquad n = 1, 2, \ldots$$

Setting

$$\pi_0 + \sum_{n=1}^{\infty} \pi_n = 1,$$

we find that

$$\pi_0 = \frac{2\mu - \lambda}{2\mu + \lambda}$$

So

$$\begin{aligned} \mathsf{E}[W] &= \textstyle\sum_{n=2}^{\infty} \mathsf{E}[W|N = n] \cdot \pi_n \\ &= \textstyle\sum_{n=2}^{\infty} \frac{n-1}{2\mu} \frac{2\mu-\lambda}{2\mu+\lambda} \frac{1}{2^{n-1}} \left(\frac{\lambda}{\mu}\right)^n \\ &= \cdots \\ &= \frac{\lambda^2}{\mu(2\mu-\lambda)(2\mu+\lambda)} \end{aligned}$$

For the M/M/2 with $\lambda = 2$ and $\mu = 1.2$, the expected waiting time for a customer to get service is

$$\frac{\lambda^2}{\mu(2\mu - \lambda)(2\mu + \lambda)} = \frac{125}{66} \approx 1.8939$$

units of time.

For the M/M/1 with $\lambda = 1$ and $\mu = 1.2$, the expected waiting time for a customer to get service is

$$\frac{\lambda}{\mu(\mu - \lambda)} = \frac{25}{6} \approx 4.16667$$

units of time.

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10. Recall that for $X \sim \Gamma(\alpha, \beta)$, $\mathsf{E}[X] = \alpha/\beta$ and $Var[X] = \alpha/\beta^2$.

In queueing theory, we used μ to denote the lifetime or service rate, not mean. So, in this case, letting S_i denote a service time, we have

$$\frac{1}{\mu} = \mathsf{E}[S_i] = 2/\nu, \qquad \sigma^2 = 2/\nu^2.$$

(a)

$$\begin{aligned} P(\text{has to wait for service}) &= P(\text{people in queue}) \\ &= 1 - P(\text{no one in queue}) \\ &= 1 - \pi_0 \\ &= 1 - (1 - \tfrac{\lambda}{\mu}) = \tfrac{\lambda}{\mu} = \tfrac{2\lambda}{\nu} \end{aligned}$$

(b)

$$\begin{aligned} L &= \frac{2\frac{\lambda}{\mu}+\lambda^2\sigma^2-\frac{\lambda^2}{\mu^2}}{2\left(1-\frac{\lambda}{\mu}\right)} \\ &= \frac{\frac{4\lambda}{\nu}+\frac{2\lambda^2}{\nu^2}-\frac{4\lambda^2}{\nu^2}}{2\left(1-\frac{2\lambda}{\nu}\right)} \\ &= \frac{\frac{2\lambda}{\nu}-\frac{\lambda^2}{\nu^2}}{1-\frac{2\lambda}{\nu}} \end{aligned}$$

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11. Let S be the service time of this typical customer. Let N be the number of customers that arrive during this service time. If S was fixed as s time units, then $N \sim Poisson(\lambda s)$. Hence,

$$\begin{aligned} P(N=n) &= \int_0^\infty P(N=n|S=s)\,\mu e^{-\mu s}\,ds \\ &= \int_0^\infty \frac{e^{-\lambda s}(\lambda s)^n}{n!}\,\mu e^{-\mu s}\,ds \\ &= \frac{\lambda^n\mu}{n!}\int_0^\infty s^n\,e^{-(\lambda+\mu)s}\,ds \end{aligned}$$

The integral is now looking like that of a gamma pdf with $\alpha = n+1$ and $\beta = \lambda+\mu$. We make this correspondence exact by putting in the appropriate constants

$$\begin{aligned} P(N=n) &= \frac{\lambda^n\mu}{n!}\int_0^\infty s^n\,e^{-(\lambda+\mu)s}\,ds \\ &= \frac{\lambda^n\mu}{n!}\frac{\Gamma(n+1)}{(\lambda+\mu)^{n+1}}\int_0^\infty \frac{1}{\Gamma(n+1)}(\lambda+\mu)^{n+1}s^n\,e^{-(\lambda+\mu)s}\,ds \\ &= \frac{\lambda^n\mu}{n!}\frac{\Gamma(n+1)}{(\lambda+\mu)^{n+1}}\cdot 1 \\ &= \frac{\lambda^n\mu}{n!}\frac{n!}{(\lambda+\mu)^{n+1}} = \frac{\lambda^n\mu}{(\lambda+\mu)^{n+1}} \end{aligned}$$

for $n = 0, 1, 2, \ldots$.

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12. Let W be the average time a customer spends in the system and let W_0 be this time excluding the service time. So,

$$W = W_0 + \frac{1}{\mu}.$$

Recall the balance equation from review problem 8:

$$L = \lambda W.$$

Then

$$L = \lambda W = \lambda(W_0 + \frac{1}{\mu}) = \lambda W_0 + \frac{\lambda}{\mu} = L_0 + \frac{\lambda}{\mu}.$$