Comments About the Conditional Dice Problem from Problem Set One

Problem 1 from Problem Set One:

An annoying friend rolls two dice and tells you that there is at least one 6. What is the probability that the sum is at least 9?

The Solution:

Out of the 36 equally likely outcomes for the two dice, there are 11 outcomes that have at least one six. Out of these 11, there are 7 outcomes where the sum is at least 9. The answer is 7/11.

An Incorrect Argument that Seems Correct at First Glance:

We know we have a six. So, we just want the probability of getting a 3, 4, 5, or 6 on a single die. This probability **is** $4/6 = 2/3$.

The problem with the incorrect argument is that it does not take into account that both dice were already thrown and information ("at least one 6") is already obtained from the pair.

Indeed, if we roll one die, get a 6, and then roll the other, the probability of getting a sum of at least nine is $4/6 = 2/3$. This is written formally as

$$P(\text{2nd roll is a 3,4,5, or 6}|\text{1st roll is a 6}) \overset{indep}{=} P(\text{2nd roll is a 3,4,5, or 6}) = 4/6 = 2/3$$

The "incorrect argument" is unintentionally putting an ordering on the rolls. (One is a 6 and now we must go on to get a 3,4,5, or 6.) Having a first and second roll is equivalent to rolling the two dice together but having labels 1 and 2 on the dice. So, we can werite the above equation as

$$P(\text{2nd die is a 3,4,5, or 6}|\text{1st die is a 6}) \overset{indep}{=} P(\text{2nd die is a 3,4,5, or 6}) = 4/6 = 2/3$$

The problem is that the "at least one 6" could have come from the 2nd die. Note that we also have

$$P(\text{1st die is a 3,4,5, or 6}|\text{2nd die is a 6}) \overset{indep}{=} P(\text{1st die is a 3,4,5, or 6}) = 4/6 = 2/3$$

Neither of these is the same as

$$P(\text{sum is at least 9}|\text{there is at least one 6})$$

which is what the question was asking.

Please see the solutions to Problem Set One for a more detailed explanation of the correct answer of 7/11.