The Monty Hall Door Problem

Monty Hall was a game show host (Let's Make a Deal). At the end of the show, he presented the contestant with three doors behind one of which was a neat-o prize. After the contestant chose a door, Monty would reveal what is behind one of the other two doors (never the big prize) and would then ask the contestant if they would like to change their mind and switch their selection to the remaining unselected door. Should you switch?

Well... not if you are the type to go home and kick yourself a lot because you had the prize all along and you gave it up.

However, you really **should switch** to have the highest probability of winning! Weird eh? Here's why... Label the doors A, B, and C. Suppose you pick door A and then Monty shows you that the prize is not behind, say, door B. At first,

$$P(\text{prize behind A}) = P(\text{prize behind B}) = P(\text{prize behind C}) = 1/3$$

so the probability that you get the prize before any of these "door switching shennangins" is 1/3.

Now Monty needs to open a door without revealing the location of the prize. If the prize **is** behind door A, then Monty will just pick another door to open for you at random since he is in no danger of opening the door with the prize:

$$P(\text{Monty opens B}|\text{prize behind A}) = P(\text{Monty opens C}|\text{prize behind A}) = 1/2$$

On the other hand,

$$\begin{aligned} P(\text{Monty opens B}|\text{prize behind B}) &= 0 \\ P(\text{Monty opens B}|\text{prize behind C}) &= 1 \end{aligned}$$

So, using that Law of Total Probability (and using the notation "o" for "Monty opens", and "p" for "prize is behind")

$$\begin{aligned} P(\text{Monty opens B}) &= P(\text{o}B|\text{p}A)P(\text{p}A) + P(\text{o}B|\text{p}B)P(\text{p}B) + P(\text{o}B|\text{p}C)P(\text{p}C) \\ &= (1/2)(1/3) + (0)(1/3) + (1)(1/3) \\ &= 1/2 \end{aligned}$$

So, the probability that you were right to begin with is (by Baye's Theorem)

$$P(\text{p}A|\text{o}B) = \frac{P(\text{o}B|\text{p}A)P(\text{p}A)}{P(\text{o}B)} = \frac{(1/2)((1/3))}{1/2} = 1/3$$

and yet the probability that the prize is actually behind door C is (again by Baye's Theorem)

$$P(\text{p}C|\text{o}B) = \frac{P(\text{o}B|\text{p}C)P(\text{p}C)}{P(\text{o}B)} = \frac{(1)(1/3)}{1/2} = 2/3$$

You have higher probability of winning the prize if you switch to door C!