APPM 4/5560

"Periodicity is a Class Property"

Periodicity is a class property. ie: If $i \leftrightarrow j$, then i and j have the same period.

Proof: To show this this, I will use the notation

$$a|b,$$

read "a divides b", to say that a divides evenly into b (or that b is a multiple of a).

Let d_i and d_j be the periods of states i and j, respectively.

Let's go...

Since $i \leftrightarrow j$, there exist integers m and n so that

$$p_{ij}^{(n)} > 0 \qquad \text{and} \qquad p_{ji}^{(m)} > 0.$$

Since

$$p_{jj}^{(m+n)} \geq p_{ji}^{(m)} p_{ij}^{(n)} > 0,$$

we must have that $d_j|(m+n)$ since, by definition of period, d_j is the greatest common divisor of all integers k such that $p_{jj}^{(k)} > 0$.

Let u be any integer such that $p_{ii}^{u} > 0$. (We know such an integer exists since one example is given by $u = n + m$ since

$$p_{ii}^{(n+m)} \geq p_{ij}^{(n)} p_{ji}^{(m)} > 0.)$$

Since

$$p_{jj}^{(m+u+n)} \geq p_{ji}^{(m)} p_{ii}^{(u)} p_{ij}^{(n)} > 0,$$

we know that $d_j|(m+k+n)$.

Now $d_j|(m+n)$ and $d_j|(m+u+n)$ implies that $d_j|u$. (Check this!)

Since u was arbitrary, we now know that d_j divides ever power k such that $P_{ii}(k) > 0$.

So, we can conclude that

$$d_i|d_j$$

since d_i is the greatest common divisor of all powers k such that $P_{ii}(k) > 0$.

A symmetric argument shows that

$$d_j|d_i.$$

If both $d_i|d_j$ and $d_j|d_i$, we must have that $d_i = d_j$.