Poisson Processes: Thinning and Superposition

Recall or show that for $X \sim \Gamma(\alpha, \beta)$, $\mathsf{E}[X] = \alpha/\beta$ and $Var[X] = \alpha/\beta^2$.

Thinning of a Poisson Process

- Suppose that $\{N(t)\}$ is a Poisson process with rate λ.
- Suppose that each event can be classified either as a "type 1" event or a "type 2" event.
- Suppose that an event is type 1 with probability p and type 2 with probability $1 - p$, independent of all other events.
- Let $N_1(t) =$ the number of type 1 events that have occurred up to time t.
- Let $N_2(t) =$ the number of type 2 events that have occurred up to time t.

Then $\{N_1(t)\}$ and $\{N_2(t)\}$ are independent Poisson processes having rates λp and $\lambda(1-p)$, respectively.

Proof:

We look at the joint distribution of $N_1(t)$ and $N_2(t)$ by conditioning on $N(t)$, the total number of events that have occurred by time t.

$$P(N_1(t) = n, N_2(t) = m) \quad = \quad \textstyle\sum_{k=0}^{\infty} P(N_1(t) = n, N_2(t) = m | N(t) = k) \cdot P(N(t) = k)$$

Since the first factor in that summand is zero when $k \neq n + m$, there is only one term to consider:

$$P(N_1(t) = n, N_2(t) = m) \quad = \quad P(N_1(t) = n, N_2(t) = m | N(t) = n + m) \cdot P(N(t) = n + m)$$

Now, given that $n + m$ events have occurred up to time t, since they are independently typed and the probability of a type 1 event is p, we will see exactly n of them be type 1 with probability

$$\binom{n+m}{n} p^n (1-p)^m.$$

So, we have that

$$\begin{aligned} P(N_1(t) = n, N_2(t) = m) &= \binom{n+m}{n} p^n (1-p)^m \cdot \tfrac{e^{-\lambda t}(\lambda t)^{n+m}}{(n+m)!} \\ &= \tfrac{1}{n!m!} p^n (1-p)^m \cdot e^{\lambda pt} \cdot e^{\lambda(1-p)t} \cdot (\lambda t)^n \cdot (\lambda t)^m \\ &= \tfrac{e^{-\lambda pt}(p\lambda t)^n}{n!} \cdot \tfrac{e^{-\lambda(1-p)t}[(1-p)\lambda t]^m}{m!} \end{aligned}$$

Since this factors into an "n" part and an "m" part, $N_1(t)$ and $N_2(t)$ are independent. The marginal distributions are

$$\begin{aligned} P(N_1(t) = n) &= \textstyle\sum_{m=0}^{\infty} P(N_1(t) = n, N_2(t) = m) \\ &= \tfrac{e^{-\lambda pt}(p\lambda t)^n}{n!} \cdot 1 \end{aligned}$$

since the second factor is the pdf of a Poisson rate $(1-p)\lambda$ distribution.

Similarly,

$$\begin{aligned} P(N_2(t) = m) &= \textstyle\sum_{n=0}^{\infty} P(N_1(t) = n, N_2(t) = m) \\ &= 1 \cdot \tfrac{e^{-\lambda(1-p)t}[(1-p)\lambda t]^m}{m!} \end{aligned}$$

(Note: To complete the proof, we have to show that $N_1(0) = 0$, $N_2(0) = 0$, and that $\{N_1\}$, and $\{N_2\}$ are processes with independent increments. I'll leave this up to you.)

- This "thinning" can easily be generalized to many arrival type occurring independently with probabilities $p_1, p_2, \ldots$

Superposition of Poisson processes

Let $\{N_1(t)\}, \{N_2(t)\}, \ldots \{N_k(t)\}$ be independent Poisson processes with rates $\lambda_1, \lambda_2, \ldots, \lambda_k$, respectively. Then $N(t) := N_1(t)+N_2(t)+\cdots+N_k(t)$ is a Poisson processes with rate $\lambda = \lambda_1+\lambda_2+\cdots+\lambda_k$.

Proof: Let's check the three Poisson axioms (definition 1).

(i) $N(0) = 0$

This is easy to verify since

$$\begin{aligned} N(0) &= N_1(0) + N_2(0) + \cdots + N_k(0) \\ &= 0 + 0 + \cdots + 0 = 0 \end{aligned}$$

since each process $N_i(t)$ is Poisson.

(ii) Independent Increments:

Consider times $0 < t_1 < t_2 < t_3 < t_4$ and consider the increments

$$N(t_2) - N(t_1), \qquad N(t_4) - N(t_3).$$

Using the fact that $N(t)$ is defined as $\sum_{i=1}^{k} N_i(t)$, these become

$$\sum_{i=1}^{k} N_i(t_2) - \sum_{i=1}^{k} N_i(t_1), \qquad \sum_{i=1}^{k} N_i(t_4) - \sum_{i=1}^{k} N_i(t_3)$$

which may be rearranged as

$$\sum_{i=1}^{k}[N_i(t_2) - N_i(t_1)], \qquad \sum_{i=1}^{k}[N_i(t_4) - N_i(t_3)].$$

Now, each term in the first sum is independent of all terms in the second sum with a different subscript since the processes $\{N_i(_t)\}$ and $N_j(t)$ are, for $i \neq j$, independent by assumption. Also, the term $N_i(t_2) - N_i(t_1)$ is independent of $N_i(t_4) - N_i(t_3)$ for all $i = 1, 2, \ldots, k$, since $N_i(t)$ is a Poisson process. Therefore,

$$\sum_{i=1}^{k}[N_i t_2 - N_i(t_1)] \qquad \text{and} \qquad \sum_{i=1}^{k}[N_i(t_4) - N_i(t_3)].$$

are independent.

(iii) $N(s+t) - N(s) \sim Poisson(\lambda t)$

To show this, we write

$$\begin{aligned} N(s+t) - N(s) &= \textstyle\sum_{i=1}^{k} N_i(s+t) - \sum_{i=1}^{k} N_i(s) \\ &= \textstyle\sum_{i=1}^{k}[N_i(s+t) - N_i(s)] \end{aligned}$$

and we note that, for $i = 1, 2, \ldots, k$ $N_i(s+t) - N_i(s) \sim Poisson(\lambda_i t)$.

We are left to show that the sum of Poisson random variables is Poisson. (Also, we want to figure out the rate parameter for this Poisson.)

For notational simplicity, I will let $X_1, X_2, \ldots, X_k$ be independent Poisson random variables with rates $\mu_1, \mu_2, \ldots, \mu_k$, respectively. (ie: X_i will represent $N_i(s+t) - N_i(s)$ and μ_i will represent $\lambda_i t$)

Let $Y = \sum_{i=1}^{k} X_i$. Then the moment generating function for Y is:

$$M_Y(t) = \mathsf{E}[e^{tY}] = \mathsf{E}\left[e^t \sum_{i=1}^{k} X_i\right] = \mathsf{E}\left[\prod_{i=1}^{k} e^{tX_i}\right] \overset{indep}{=} \prod_{i=1}^{k} \mathsf{E}[e^{tX_i}] = \prod_{i=1}^{k} M_{X_i}(t)$$

$$\overset{?}{=} \prod_{i=1}^{k} \exp\left[\mu_i(e^{t-1})\right] = \exp\left[(\sum_{i=1}^{k} \mu_i)(e^t - 1)\right]$$

(?: This is the mgf for a Poisson random variable with rate μ_i. Please verify this.)

The final expression is the mgf for a Poisson random variable with rate $\sum_{i=1}^{k} \mu_i$. Therefore,

$$X_1, X_2, \ldots, X_k \text{ indep. with } X_i \sim Poisson(\mu_i) \;\Rightarrow\; \sum_{i=1}^{k} X_i \sim Poisson(\sum_{i=1}^{k} \mu_i)$$

So,

$$N_i(s+t) - N_i(s) \sim Poisson(\lambda_i t) \qquad \Rightarrow N(t) = \sum_{i=1}^{k}[N_i(s+t) - N_i(s)] \sim Poisson((\sum_{i=1}^{k} \lambda_i)t)$$

Since all three Poisson axioms are satisfied, we have that $N(t)$ is a Poisson process with rate $\sum_{i=1}^{k} \lambda_i$.

Bam! Thanks for reading.