Unfinished Example from Class: A Sum of Uniforms

Let X and Y be independent random variables each distributed uniformly on the interval from 0 to 1. ($X, Y \overset{iid}{\sim} unif(0,1)$) Find the pdf of $Z = X + Y$.

We will begin by finding the cdf of Z and then taking the derivative in order to find the pdf. Note that Z can take on values between 0 and 2.

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P(X + Y \leq z) \\ &= \textstyle\int_{-\infty}^{\infty} P(X + Y \leq z | Y = y) \cdot f_Y(y)\, dy \end{aligned}$$

Since $f_Y(y)$ is 1 on the interval $(0, 1)$ and zero otherwise, this becomes

$$\int_0^1 P(X + Y \leq z | Y = y)\, dy.$$

We plug in y for Y

$$\int_0^1 P(X \leq z - y | Y = y)\, dy$$

and use the fact that X and Y are independent

$$= \int_0^1 P(X \leq z - y)\, dy$$

Since X is uniform on $(0, 1)$,

$$P(X \leq x) = \begin{cases} 0 & , \quad x < 0 \\ x & , \quad 0 \leq x < 1 \\ 1 & , \quad x \geq 1 \end{cases}$$

Case: $1 < z < 2$:

For a fixed z between 1 and 2, then $0 < y < 1$ implies that $z - y$ is always above 0 but may also be above 1. So,

$$P(X \leq z - y) = \begin{cases} z - y & , \quad \text{if } z - y \leq 1 \\ 1 & , \quad \text{if } z - y > 1 \end{cases} = \begin{cases} z - y & , \quad \text{if } y \geq z - 1 \\ 1 & , \quad \text{if } y < z - 1 \end{cases}$$

Since $0 < z - 1 < 1$, we have

$$\begin{aligned} F_Z(z) &= \textstyle\int_0^1 P(X \leq z - y)\, dy = \int_0^{z-1} 1\, dy + \int_{z-1}^1 (z - y)\, dy \\ &= z - 1 + z - \tfrac{1}{2}z^2 = 2z - \tfrac{1}{2}z^2 - 1 \end{aligned}$$

So, the pdf of F for $1 < z < 2$ is

$$f_Z(z) = \frac{d}{dz}\left[2z - \frac{1}{2}z^2 - 1\right] = 2 - z.$$

Case: $0 < z < 1$**:**

For a fixed z between 0 and 1, then $0 < y < 1$ implies that $z - y$ is always below 1 but may also be below 0. So,

$$P(X \leq z - y) = \begin{cases} z - y & , \text{ if } z - y \geq 0 \\ 0 & , \text{ if } z - y < 0 \end{cases} = \begin{cases} z - y & , \text{ if } y \leq z \\ 1 & , \text{ if } y > z \end{cases}$$

Since $0 < z < 1$, we have

$$\begin{aligned} F_Z(z) &= \textstyle\int_0^1 P(X \leq z - y)\,dy = \int_0^z (z - y)\,dy + \int_z^1 0\,dy \\ &= \tfrac{1}{2}z^2 + 0 = \tfrac{1}{2}z^2 \end{aligned}$$

So, the pdf of F for $0 < z < 1$ is

$$f_Z(z) = \frac{d}{dz}\frac{1}{2}z^2 = z.$$

Pulling it all together now, the pdf for Z is

$$f_Z(z) = \begin{cases} z & , \text{ if } 0 \leq z \leq 1 \\ 2 - z & , \text{ if } 1 \leq z \leq 2 \\ 0 & , \text{ otherwise} \end{cases}$$