Midterm Exam

2 March 2005

- Name:
- Code Name:
  (by listing a short code-name, you're giving the instructor permission to post your homework and exam scores under this name)
- Calculators allowed.
- Show all work and box in answers where appropriate.
- Be sure to turn in *this sheet* with your other work.
- There are 100 points on the exam.
- **BUDGET YOUR TIME!!!**
- **READ THE PROBLEMS CAREFULLY!!**

Part 1: True or False (circle your answer; 4 Points Each)

(a) True or False: A predictor-corrector scheme such as AB3/AM2 is an explicit scheme.

(b) True or False: An n^{th}-order Runge-Kutta method applied to an IVP with step-size h has local truncation order that is $O(h^n)$.

(c) True or False: The primary difficulty associated with using non-orthogonal polynomials in a continuous least-squares approximation is that one must solve a nonlinear system of equations.

(d) True or False: The Gram-Schmidt Process applied to the interval $[-1, 1]$ with $w(x) = x$ yields Legendre Polynomials.

Part 2: Initial-Value Problems

(a) (6 Points) Without the use of equations, explain what is meant by a **stiff** ODE.

(b) (10 Points) Consider $f(t, y) = -4y + \cos(yt^2)$. Does $f(t, y)$ satisfy a Lipschitz condition on $D = \{(t, y) | -10 \leq t \leq 10, -\infty < y < \infty\}$? If so, what is the Lipschitz constant?

(c) (8 Points) Classify the following ODE below. Write the ODE in a form such that we could find an approximate solution for $y(t)$ using one of the methods discussed in class. (You don't need to solve the equation.)

$$\frac{\mathrm{d}^3 y}{\mathrm{d}t^3} + t^2 \cos\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right) = t^2, \quad y(0) = 0$$

(d) (18 Points) Use the midpoint method (RK2) and $\Delta t = 0.1$ to find y_2 given $y_1 = 15$, for the following IVP. Find an upper bound for Δt for the midpoint method for a stable solution.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -100y,$$

Part 3: Approximation Theory

(a) (14 Points) Consider the following first two terms of a of a series of polynomials. Are $\phi_0(x)$ and $\phi_1(x)$ orthogonal on $[0, 1]$ with respect to $w(x) = x$? Be sure to show your work.

$$\phi_0(x) = 1, \quad \phi_1(x) = x - \frac{2}{3}$$

(b) (8 Points) Approximate the following function using the first two terms in the above series by minimizing the square of the error.

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 0.5 \\ 0 & \text{for } 0.5 < x \leq 1 \end{cases}$$

(c) (14 Points) Find the general continuous least squares trigonometric polynomial $S_n(x)$ for $g(x) = x$, which is periodic on $x \in (-\pi, \pi)$. ($g(x)$ is called a sawtooth function)

(d) (6 Points) What do we mean by 'Fast' when we refer to a Fast Fourier Transform?

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PART 1

(a) T (b) F (c) F (d) F

PART 2

(a) STIFFNESS OCCURS IN A PROBLEM WHEN THERE ARE TWO VERY DIFFERENT SCALES OF THE INDEPENDENT VARIABLE(S) ON WHICH THE DEPENDENT VARIABLES ARE CHANGING

(b) $f(t,y) = -4y + \cos(yt^2)$ $D = \{(t,y) \mid -10 \le t \le 10, -\infty < y < \infty\}$

$$\left|\frac{\partial f}{\partial y}\right| = \left|-4 - t^2 \sin(yt^2)\right| \le 4 + t^2 \le 104$$

$\boxed{\text{YES, } f(t,y) \text{ SATISFIES A LIPSCHITZ CONDITION AND } L = 104}$

(c) 3RD-ORDER, NONLINEAR, NON-CONSTANT COEF., NON HOMOGENEOUS ODE

LET $y_1 = y$, $y_2 = \frac{dy_1}{dt} = \frac{dy}{dt}$, $y_3 = \frac{dy_2}{dt} = \frac{d^2y_1}{dt^2} = \frac{d^2y}{dt^2}$

$$\boxed{\frac{dy_1}{dt} = y_2 \qquad \frac{dy_2}{dt} = y_3, \qquad \frac{dy_3}{dt} = -t^2\cos(y_2) + t^2}$$

(d) MIDPOINT METHOD $y_{i+1} = y_i + h f(t_i + h/2, \alpha_i)$; $\alpha_i = y_i + \frac{h}{2} f(t_i, y_i)$

FOR $h = 0.1$; $y_1 = 15$ $y'(t) = -100y$

$\alpha_1 = 15 + 0.05(-100 \cdot 15) = -60$

$y_2 = y_1 + 0.1(-100 \cdot \alpha_1) = \boxed{615 = y_2}$

CONSIDER $y'(t) = -\lambda y$ $(\lambda = 100)$

$\alpha_i = y_i + \frac{h}{2}(-\lambda y_i)$

$y_{i+1} = y_i + h(-\lambda)\left(y_i - \frac{h}{2}(\lambda y_i)\right) = y_i\left(1 - \lambda h + \frac{1}{2}h^2\lambda^2\right)$

REQUIRE $\left|1 - \lambda h + \frac{1}{2}h^2\lambda^2\right| < 1$ $(\lambda > 0)$

$\frac{1}{2}\lambda^2 h^2 < \lambda h \Rightarrow h < 2/\lambda$ $\boxed{h < 2/100}$

PART 3

(a) $\int_0^1 x\,\phi_0(x)\,\phi_0(x)\,dx = \int_0^1 x\,dx = \boxed{1/2}$

$$\int^1 x\,\phi_1(x)\,\phi_0(x)\,dx = \int_0^1 \left(x^2 - \frac{2}{3}x\right)dx = \left[\frac{x^3}{3} - \frac{x^2}{3}\right]_0^1 = \frac{1}{3} - \frac{1}{3} = \boxed{0}$$

$$\int_0^1 x\,\phi_1(x)\,\phi_1(x)\,dx = \int_0^1 x\left(x^2 - \frac{4}{3}x - \frac{4}{9}\right)dx = \left[\frac{x^4}{4} - \frac{4}{9}x^3 - \frac{2x^2}{9}\right]_0^1 = \frac{1}{4} - \frac{2}{9} = \boxed{\frac{1}{36}}$$

YES, THEY'RE ORTHOGONAL

(c) $a_k = 0$ FOR ALL K ($g(x)$ IS AN ODD FUNCTION)

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)\,dx = \frac{1}{\pi}\left\{\left[-\frac{x}{k}\cos(kx)\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{1}{k}\cos(kx)\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-2\frac{\pi}{k}(-1)^k + \left[\frac{1}{k^2}\overset{0}{\cancel{\sin(kx)}}\right]_{-\pi}^{\pi}\right\} = \frac{2(-1)^{k+1}}{k}$$

$$\boxed{S_n(x) = \sum_{k=1}^{n-1}\frac{2(-1)^{k+1}}{k}\sin(kx)}$$

(b) $S(x) = a_0\,\phi_0(x) + a_1\,\phi_1(x)$

$$a_0 = 2\int_0^{1/2} x\,dx = \left[\frac{x^2}{2}\right]_0^{1/2} = 1/4$$

$$a_1 = 36\int_0^{1/2}\left(x^2 - \frac{2}{3}x\right)dx = 36\left[\frac{x^3}{3} - \frac{x^2}{3}\right]_0^{1/2} = 12\left(\frac{1}{8} - \frac{2}{8}\right) = -\frac{3}{2}$$

$$\boxed{S(x) = \frac{1}{4} - \frac{3}{2}\left(x - \frac{2}{3}\right)}$$

(d) A NAIVE FOURIER TRANSFORM FOR $2m$ DATA REQUIRES $O(4m^2)$ OPERATIONS; AN FFT REQUIRES $O(m\log_2 m)$ OPERATIONS. (FOR $2m = 2^p$)