PARTIAL DIFFERENTIAL EQUATIONS MIDTERM EXAMINATION
November 2006

You have 2 1/2 hours to complete this exam. You are the timekeeper. Please indicate your start and end times. Each problem is worth 25 points. Please start each problem on a new page. A sheet of convenient formulae is attached.

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1. a) Solve the initial value problem and determine the values of x, y for which it exists

$$u_x + u^2 u_y = 1, \qquad u(x,0) = 1$$

b) Solve the initial value problem and determine the values of x, y, z for which it exists

$$x u_x + y u_y + u_z = u, \qquad u(x,y,0) = h(x,y)$$

2. Discuss the following questions for each of the Fourier series (a-e). Explain your answer by quoting the relevant theorems. You need not prove any theorems.

(i) What do our theorems for convergence tell us about these series (discuss uniform convergence, pointwise convergence and convergence in the mean)?

(ii) If the series converges, to what type of function does it converge (i.e. what is the most restrictive set among $L_1, L_2, D^0, D^1, C^0, C^k$, that you know the function belongs to)?

(iii) Can the series be differentiated term-by-term (i.e. if you differentiate the sum is it equal to the series obtained by differentiating term-by-term?

a) $\sum_{n=1}^{\infty} \frac{1}{1+n^{10}} \sin(nx)$ *b)* $\sum_{n=1}^{\infty} (-1)^n \frac{\cos(nx)}{n^{2/3}}$ *c)* $\frac{1}{2} \sum_{n=1}^{\infty} [e^{-n^2} \sin(nx) + e^{-2n} \cos(nx)]$

d) $\sum_{n=1}^{\infty} \frac{n}{1+n^2} \sin(nx)$ *e)* $\sum_{n=1}^{\infty} \frac{\cos(2nx)}{\log n}$

3. Let $f(x) = 1 - |x|$.

a) Find the Fourier series for f on the interval $[-\pi, \pi]$.

b) Does the series converge? Does it converge uniformly? If so where?

c) Sketch the function that the Fourier series converges to.

d) State Parseval's relation. Under what conditions does it apply?

e) Does Parseval's relation apply to f? If so write out the relation for f.

4. Consider the equation

$$\begin{aligned} u_t + bu = \kappa u_{xx} \quad 0 < x < \pi, \quad t > 0, \quad b, \kappa > 0 \\ \left.\begin{aligned} u(0,t) = 0 \\ u(\pi,t) = 0 \end{aligned}\right. \qquad t > 0 \\ u(x,0) = x(\pi - x) \quad 0 < x < \pi. \end{aligned}$$

a) Obtain a formal solution to this equation. (You do not need to compute the Fourier coefficients, just set up the integrals).

b) (**You can do this without doing part a**). Discuss the validity of your solution. Does the series converge? If so where?