① $u_x + u^2 u_y = 1 \qquad u(x,0) = 1$

$$\left.\begin{aligned} \frac{dx}{d\tau} &= 1 \\ \frac{dy}{d\tau} &= z^2 \\ \frac{dz}{d\tau} &= 1 \end{aligned}\right\} + \text{I-C} \quad (s,0,1) : J = \begin{vmatrix} x_s & y_s \\ x_\tau & y_\tau \end{vmatrix}_{\tau=0} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

$(s,0,1)$ non characteristic.

$$\left.\begin{aligned} x &= \tau + s \\ z &= \tau + 1 \end{aligned}\right\} \Rightarrow y = \frac{1}{3}\left((\tau+1)^3 - 1\right)$$

$$\Rightarrow z = (3y+1)^{1/3} \Rightarrow u(x,y) = (3y+1)^{1/3} \text{ exist } \forall (x,y).$$

② $x u_x + y u_y + u_z = u \quad ; \quad (s, \mu, \tau)$

$$\left.\begin{aligned} \frac{dx}{d\tau} &= x \\ \frac{dy}{d\tau} &= y \\ \frac{dz}{d\tau} &= 1 \\ \frac{dw}{d\tau} &= w \end{aligned}\right\} + \text{I-C} \quad (s, \mu, 0, h(s,\mu))$$

$$J = \begin{vmatrix} x_s & y_s & z_s \\ x_\mu & y_\mu & z_\mu \\ x_\tau & y_\tau & z_\tau \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s & \mu & 1 \end{vmatrix} = 1 \neq 0$$

$(s, \mu, 0, h(s,\mu))$ non-char.

$$\left.\begin{aligned} x &= s e^\tau \\ y &= \mu e^\tau \\ z &= \tau \\ w &= h(s,\mu) e^\tau \end{aligned}\right\} \Rightarrow w = h(x e^{-z}, y e^{-z}) e^z .$$

2(a) $\sum_{n=1}^{\infty} \frac{1}{1+n^{10}} \sin(nx)$.

(i) Converges uniformly + absolutely since $\sum_1 \left|\frac{1}{1+n^{10}}\right|$ is convergent.

(ii). $f \in C^8$

(iii) Yes can be differentiated term by term.

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{\cos(nx)}{n^{2/3}}$

(i) Converges in the mean, $\sum_1 \frac{1}{n^{4/3}}$ converges

Doesnot converge pointwise at $x = \pi, \dots \quad (-1)^n \cos(n\pi) = 1$.

(ii) $f \in L_2$.

(iii) No cannot be differentiated term by term!

(c) $\frac{1}{2} \sum_{n=1}^{\infty} e^{-n^2} \sin nx + e^{-2n} \cos nx$

(i) unif + abs conv.

(ii) C^{∞}

(iii) Yes.

(d) $\sum_{n=1}^{\infty} \frac{n}{1+n^2} \sin nx$.

(i) Converges in mean $\sum_{n=1}^{\infty} \frac{n^2}{(1+n^2)^2} < \infty$.

series pointwise convergent.

$\sum \left|\frac{n}{1+n^2}\right| \to \infty$ not uniformly conv.

(ii) $f \in D^1$ (iii) No.

(e) $\sum_{n=1}^{\infty} \frac{\cos(2nx)}{\log n}$ not convergent.

(i), (ii) (iii).

3.) Let $f(x) = 1 - |x|$

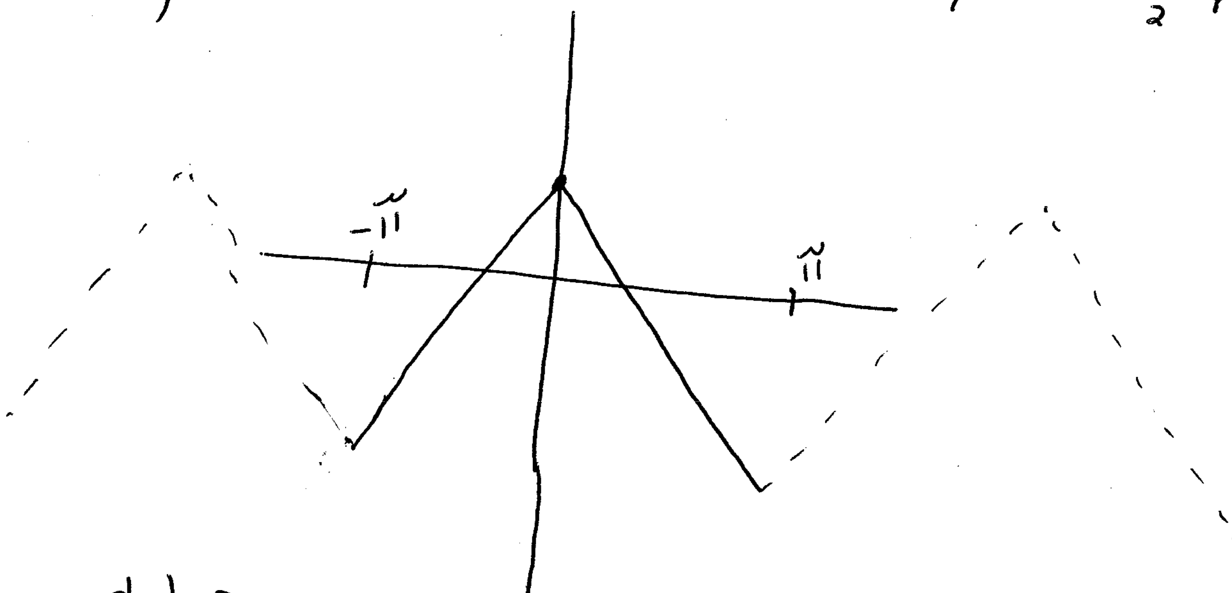
a.) Find the Fourier series for f on the interval $[-\pi, \pi]$.

$$F.S. f(x) = 1 - F.S. |x| = 1 - \left(\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi} \frac{\cos(n\pi) - 1}{n^2} \cos(nx)\right) = \frac{2-\pi}{2} - \sum_{n=1}^{\infty} \frac{4}{\pi} \frac{(-1)^{2n-1} \cos}{(2n-1)^2}$$

$$= \frac{2-\pi}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2}$$

b.) The series converges uniformly

c.)



d.) and e.) Parseval's relation

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\, dx = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

- As it refers to this problem,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\, dx = \frac{(2-\pi)^2}{2} + \sum_{n=1}^{\infty} \left(\frac{4}{\pi (2n-1)^2}\right)^2$$

- Under what conditions does it apply?

If $f(x)$ is 2π periodic and continuous

- This relation applies to $1 - |x|$
- Integrating and expanding gives

$$\pi^4 = 96\left(1 + \frac{1}{81} + \frac{1}{625} + \dots\right)$$

4.) Consider the eqn.

$$U_t + bU = kU_{xx} \quad , 0 < x < \pi \; , t > 0 \quad , b, k > 0$$

$$U(0,t) = 0$$

$$U(\pi, t) = 0$$

$$U(x,0) = x(\pi - x)$$

- Use separation of variables

$$U(x,t) = F(x)G(t)$$

$$F(x)\dot{G}(t) + bF(x)G(t) = kF''(x)G(t)$$

$$\Rightarrow \frac{\dot{G}(t)}{G(t)} + b = \frac{kF''(x)}{F(x)} = \lambda$$

- Solve for $F(x)$

$\lambda = 0$ yields trivial sol.

- $\lambda < 0$, Let $\lambda = -p^2$

$$F''(x) + \frac{p^2}{k}F(x) = 0$$

$$\Rightarrow \quad F(x) = A\cos\left(\frac{p}{\sqrt{k}}x\right) + B\sin\left(\frac{p}{\sqrt{k}}x\right)$$

$$F(0) = A = 0$$

$$F(\pi) = B\sin\left(\frac{p}{\sqrt{k}}\pi\right) = 0$$

$$\Rightarrow \frac{p\pi}{\sqrt{k}} = n\pi \Rightarrow p_n = n\sqrt{k}$$

- $\lambda > 0$ yields trivial sol.

$$F_n(x) = B_n \sin\left(\frac{p_n}{\sqrt{k}}x\right) = B_n \sin(nx)$$

- Solve for $G(t)$

$$\dot{G}(t) + bG(t) + n^2 k\, G(t) = 0$$

$$\dot{G}(t) + (b + n^2 k)G(t) = 0$$

$$G(t) = e^{-(b+n^2k)t} \cdot C_n$$

$$U(x,t) = \sum_{n=1}^{\infty} e^{-(b+n^2k)t} \sin(nx)\, B_n$$

- Initial value

$$U(x,0) = \sum_{n=1}^{\infty} B_n \sin(nx) = x(\pi - x)$$

$$\Rightarrow B_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(nx)\,dx$$

$$U(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) e^{-(b+n^2k)t}$$

b.) This series converges uniformly and the solution exists $\forall\ t > 0$.