

BASIC FACTS ABOUT LINEAR O.D.E.S

1. Definitions:

a) A linear ordinary differential equation (ODE) of order N has the form

$$p_0(t) \frac{d^N y}{dt^N} + p_1(t) \frac{d^{N-1} y}{dt^{N-1}} + \dots + p_N(t) y = r(t). \quad (\text{I})$$

The coefficients, $\{p_0(t), p_1(t), \dots, p_N(t), r(t)\}$, may each depend on (t) . We will assume *either* that these coefficients are continuous and single-valued on a finite real interval, $t_0 \leq t \leq t_1$, *or* that they are meromorphic (*i.e.*, analytic except for poles, each of finite order) for complex t , $|t| < \infty$.

b) The equation is homogeneous if $r(t) = 0$, inhomogeneous otherwise.

c) Any point (t) at which the coefficients are analytic *and* $p_0(t) \neq 0$ is an ordinary point of (I). If the coefficients are meromorphic, then other points are called singular points or singularities

d) A differential equation is a purely local statement, relating various quantities at each point, t . To solve a differential equation is to infer nonlocal (or global) information from it; *e.g.*, given enough information at t_0 , find $y(t_1)$, for $t_1 \neq t_0$.

2. The solutions of a linear homogeneous ODE form a linear vector space.

a) Let $\{y_1(t), y_2(t), \dots, y_k(t)\}$ denote k solutions of a linear homogeneous ODE. These functions are linearly dependent in an interval $t_0 \leq t \leq t_1$ if there exist constants $\{c_1, \dots, c_k\}$, not all zero, such that

$$c_1 y_1(t) + c_2 y_2(t) + \dots + c_k y_k(t) = 0, \quad t_0 \leq t \leq t_1.$$

Otherwise they are linearly independent.

b) A linear homogeneous ODE of order N has exactly N solutions that are linearly independent. Usually the simplest way to show that N specific solutions are linearly independent at t is to show that their Wronskian does not vanish there.

c) The Wronskian of k differentiable functions is

$$W(t) = \det \begin{bmatrix} u_1(t) & u_2(t) & \dots & u_k(t) \\ u_1'(t) & u_2'(t) & \dots & u_k'(t) \\ \vdots & \vdots & & \vdots \\ u_1^{(k)}(t) & u_2^{(k)}(t) & \dots & u_k^{(k)}(t) \end{bmatrix}. \quad (\text{II})$$

The functions are linearly independent at $t = \tau$ if their Wronskian does not vanish there (*i.e.*, $W(\tau) \neq 0$).

d) The set of all solutions of a linear homogeneous ODE form an N -dimensional vector space. Any set of N linearly independent solutions are basis vectors that span the space.

e) The Wronskian of N solutions measures the volume of an element of the N -dimensional phase space, and its vanishing at t means (geometrically) that the volume element has collapsed to something of lower dimension.

f) The Wronskian, W , of N solutions of a linear homogeneous ODE satisfies a simple ODE:

$$p_0 \frac{dW}{dt} + p_1 W = 0. \quad (\text{III})$$

Suppose $W(t) \neq 0$ at $t = \tau$. Then two results follow from (III).

- (i) If $[p_1(t)/p_0(t)]$ is finite along a smooth path from τ to \mathbf{T} , then (III) shows that $W(t)$ cannot vanish along this path and the solutions must be linearly independent all along this path.
- (ii) If $[p_1(t)/p_0(t)]$ vanishes all along the path, then $W(t)$ is a constant of the motion, and it constrains the possible behavior of the solutions of the ODE.

3. If a linear, homogeneous equation has constant coefficients, then one may seek exponential solutions:

$$y(t) = Ae^{mt}. \quad (\text{IV})$$

Then (m) satisfies an N^{th} order algebraic equation, and each of the N algebraic roots corresponds to a solution of the ODE. If the N roots are distinct, then each root represents a distinct solution of the ODE; they are linearly independent. The behaviour (in t) of each solution of the ODE depends qualitatively on whether its value of (m) is real, pure imaginary or neither, and whether or not it is a repeated root.

4. Analytic solutions of linear homogeneous ODEs

a) In the neighborhood of an ordinary point ($t=\tau$) of a linear homogeneous ODE, solutions can be constructed by substituting in a formal power series,

$$y(t) = \sum_{n=0}^{\infty} a_n (t-\tau)^n,$$

then determining the coefficients recursively. A function so constructed is analytic within the radius of convergence of the series.

b) The radius of convergence is the distance in the complex t -plane from (τ) to the nearest singularity of a coefficient function, including a zero of $p_0(t)$. If the coefficient functions are entire (no singularities for $|t| < \infty$), then so are the solutions of the equation.

c) The solutions of a linear, homogeneous ODE have no singularities except at singular points of the coefficient functions. These singularities are called fixed, because their locations are fixed by the equation itself. [This property holds only for linear equations. A very simple nonlinear equation,

$$\frac{dw}{dt} + w^2 = 0,$$

has as its general solution: $w(t) = \frac{1}{(t-t_0)}$. Here t_0 is the constant of integration, but it also locates the singularity.]

d) An analytic solution often can be extended beyond its radius of convergence, using analytic continuation.

5. Variation of Parameters

a) Variation of parameters is a standard method to find the solution of a linear, nonhomogeneous equation like (I). The method begins with N linearly independent solutions of the homogeneous ODE, from which it produces the solution of the nonhomogeneous ODE in the form of specific integrals (that perhaps cannot be evaluated in closed form).

b) The method also can be used to solve equations like the (time-independent) Schrödinger equation:

$$\frac{d^2 u}{dx^2} + k^2 u = q(x)u, \quad (\text{V})$$

where $q(x)$ is a known but possibly complicated function. One thinks of $[q(x)u]$ as known, then the method replaces (V) and its boundary conditions with an integral equation, which must then be solved. If (V) is given with initial data $[u(x_0) = c_1, u_x(x_0) = c_2]$, then the integral operator is of Volterra type, and the integral equation can be solved by iteration.

6. The theory of linear difference equations is similar to that of differential equations. The discrete index (n) plays the role of the continuous variable (t).

a) A second-order linear difference equation has the form

$$p_n y_{n+1} + q_n y_n + r_n y_{n-1} = s_n, \quad n = 1, 2, \dots$$

where $\{p_n, q_n, r_n, s_n\}$ are known sequences of numbers.

b) The equation is homogeneous if $s_n = 0$.

c) A second-order homogeneous difference equation has two linearly independent solutions. Consider two solutions $\{x_n, y_n\}_{n=1}^{\infty}$. They are linearly independent at n if their Wronskian,

$$W_n = x_n y_{n+1} - x_{n+1} y_n,$$

is non-zero at n .

d) The simplest situation is a homogeneous equation with constant coefficients:

$$p y_{n+1} + q y_n + r y_{n-1} = 0. \quad (\text{VI})$$

The analogue of the exponential ansatz ($y(t) = Ae^{mt}$) is

$$y_n = Az^n.$$

Substituting this into (VI) yields a quadratic equation for (z) , whose two roots represent the two solutions of (VI). For any particular value of z , z^n is an entire function of n .