APPM 5600 NUMERICAL ANALYSIS

FINAL EXAM

TIME: 55 MINUTES December 11, 2002, 9:00 - 9:50 a.m.

You must complete problems 1 and 2. Choose two problems from the set {3, 4, 5}. Put an X through the problem you do not want to be graded. I will only grade 4 problems, so do not forget to mark one out. No aids except calculators permitted

NAME:_____

For Grader Only	
1	/ 25
2	/ 25
3	/ 25
4	/ 25
5	/ 25
Σ	/100

1. a. Given the matrix

$$A = \begin{bmatrix} 1 & -3 \\ 1 & 3 \\ 0 & 4 \\ -1 & 1 \\ 1 & 1 \end{bmatrix},$$

construct the components of the Householder reflection, $H = I - \beta \underline{x}\underline{x}^t$, that reflects the first column of A onto the first coordinate axis.

b. Then, compute HA.

c. What are the singular values of A?

Answer: a.) We seek $H = I - \beta \underline{x}\underline{x}^t$. Write the columns of A as $\underline{a}_1$, $\underline{a}_2$. We have $\|\underline{a}_1\| = 2$. Set

$$\underline{x} = \begin{pmatrix} 1+2 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

We know

$$< \underline{x}, \underline{x} >= 2\|\underline{a}_1\|(\|\underline{a}_1\| + |a_{11}|) = 4(2+1) = 12$$

Set $\beta = \frac{2}{<\underline{x}, \underline{x}>} = \frac{1}{6}$.

b. Thus,

$$HA = (I - \frac{1}{6}\underline{x}\underline{x}^t)A = A^{(2)} = \begin{bmatrix} -2 & 0 \\ 0 & 4 \\ 0 & 4 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}$$

c. The singular values of A are the found from the eigenvalues of

$$A^*A = (A^{(2)})^* H^* H A^{(2)} = \begin{bmatrix} 4 & 0 \\ 0 & 36 \end{bmatrix}.$$

Thus the singular values of A are $\sigma_1 = 6$ and $\sigma_2 = 2$.

2. Let

$$A = S \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix} S^{-1},$$

where S is nonsingular.

a. What is the characteristic polynomial of A?

b. What is the minimal polynomial of A?

c. Consider the shifted inverse power method using the matrix A above:

$$\begin{aligned} \underline{z}_n &= (A - 5I)^{-1}\underline{y}_{n-1} \\ \underline{y}_n &= \frac{\underline{z}_n}{\|\underline{z}_n\|} \\ \alpha_n &= \frac{\langle A\underline{y}_n, \underline{y}_n\rangle}{\langle \underline{y}_n, \underline{y}_n\rangle}. \end{aligned}$$

Will the sequence $\underline{y}_n$ converge and, if so, to what? (be specific)

d. Will the sequence $\{\alpha_n\}$ converge and, if so, to what? (be careful)

e. If it converges, at what rate will it converge?

Answer:

a. The characteristic polynomial of A is;

$$p_C(\lambda) = (\lambda - 2)^3(\lambda - 4)(\lambda - 7)^2.$$

b. The minimal polynomial of A is:

$$p_M(\lambda) = (\lambda - 2)^2(\lambda - 4)(\lambda - 7)^2.$$

c. The vectors $\underline{y}_n$ will converge to the eigenvector associated with the largest eigenvalue of $(A - 5I)^{-1}$. The eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 4$, and $\lambda_3 = 7$. The eigenvalues of $(A - 5I)^{-1}$ are

$$\mu_1 = \frac{1}{\lambda_1 - 5} = -\frac{1}{3}, \qquad \mu_2 = \frac{1}{\lambda_2 - 5} = -1, \qquad \mu_3 = \frac{1}{\lambda_3 - 5} = \frac{1}{2}.$$

The largest in absolute value is $|\mu_2| = 1$. Thus, $\underline{y}_n$ will converge to the eigenvector corresponding to eigenvalue μ_2. That is,

d. α_n will converge to the corresponding eigenvalue of A, that is, $\alpha_n \to \lambda_2 = 4$.

e. The rate of convergence will be the ratio of the second largest eigenvalue of $(A - 5I)^{-1}$ to the largest eigenvalue of $(A - 5I)^{-1}$, which yields:

$$\frac{|4 - \alpha_n|}{|4 - \alpha_{n-1}|} \to \frac{|\mu_3|}{|\mu_2|} = \frac{1}{2}.$$

3. a. Determine the values of a, b, c so that the following is a cubic spline with knots at 0,1,2:

$$s(x) = \begin{cases} 3 - 2x + 2x^3 & x \in [0,1] \\ a + b(x-1) + c(x-1)^2 + d(x-1)^3 & x \in [1,2] \end{cases}$$

b. Determine d that minimizes $\int_0^2 (s''(x))^2 dx$.

c. Is there a value for d that makes $s(x)$ a natural spline? Explain your answer.

Answer: a. We must choose a, b, c so that $s(x)$, $s'(x)$, and $s''(x)$ are continuous at $x = 1$. This yields the equations

$$\begin{aligned} s(1^-) &= 3 = s(1^+) = a \\ s'(1^-) &= 4 = s'(1^+) = b \\ s''(1^-) &= 12 = s''(1^+) = 2c \end{aligned}$$

Thus,

$$s(x) = \begin{cases} 3 - 2x + 2x^3 & x \in [0,1] \\ 3 + 4(x-1) + 6(x-1)^2 + d(x-1)^3 & x \in [1,2] \end{cases}$$

b. The choice of d cannot effect $\int_0^1 (s''(x))^2$, so we choose d to minimize

$$\begin{aligned} \int_1^2 (s''(x))^2 &= \int_1^2 (12 + 6d(x-1))^2 = 36 \int_0^1 (2 + dz)^2 \\ &= 36 \int_0^1 (4 + 4dz + d^2 z^2) = 36(4 + 2d + \frac{d^2}{3}) \end{aligned}$$

Minimizing over d yields $d = -3$.

c. The value $d = -2$ yields $s''(2) = 0$. Since, $s''(0) = 0$, this value of d would yield a natural spline.

4. Let $x_i = x_0 + ih$, for $i = 1, \ldots, n$ be a mesh of equally spaced points. Let $f(x) \in C^{(n+2)}[x_0, x_n]$ and let $p_n(x)$ be the unique polynomial of degree n that interpolates $f(x)$ on $x_0, \ldots, x_n$.

   a. Give an expression for the interpolation error, $\mathcal{E}_n(f) = f(x) - p_n(x)$, in terms of both (i.) divided differences of $f(x)$ and (ii.) derivatives of $f(x)$.

   b. Give an expression for the error in the Newton-Cotes formula, $E_n(F) = \int_{x_0}^{x_n} (f(t) - p_n(t))dt$, in terms of the mesh space $h = (x_n - x_0)/n$ and derivatives of $f(x)$.

   c. What is the degree of precision of the Newton-Cotes formula?

   d. Consider using a **composite** Simpson's rule to integrate $f(x) = e^x$ on the interval $[0, 1]$. Each time the mesh spacing h is halved, how would you expect the error to be affected? If the function were $f(x) = \sqrt{x}$ what would you expect? Explain your answer.

Answer:

a. The interpolation error is given by

$$\mathcal{E}_n(f) = f(t) - p_n(t) = \Pi_{j=0}^{n}(t - x_j) f[x_0, x_1, \ldots, x_n, t] = \Pi_{j=0}^{n}(t - x_j) \frac{f^{(n+1)}(\zeta)}{(n+1)!}$$

for some $\zeta(t) \in (x_0, x_n)$.

b. The integration error is given by

$$E_n = \int_{x_0}^{x_n} (f(t) - p_n(t))dt = \begin{cases} C_n h^{n+2} \frac{f^{(n+1)}(\zeta)}{(n+1)!} & \text{if } n \text{ odd} \\ C_n h^{n+3} \frac{f^{(n+2)}(\zeta)}{(n+2)!} & \text{if } n \text{ even} \end{cases}$$

c. The degree of precision is:

$$\text{DOP} = \begin{cases} n & \text{if } n \text{ odd} \\ n+1 & \text{if } n \text{ even} \end{cases}$$

d. The error in **composite** Simpson's rule is

$$E_n(f) = -\frac{h^4(b-a)}{180} f^{(4)}(\zeta) \quad \zeta \in (a, b).$$

Thus, the error will be approximately reduced by a factor of $1/16$ for $f(x) \in C^{(4)}(0, 1)$. For $f(x) = \sqrt{x}$, we see $f(x) \notin C^{(4)}(0, 1)$ so the factor will be much larger than $1/16$.

5. Which of the following iterations will converge to the indicated fixed point(provided x_0 is sufficiently close to α). If it does converge, give the order of convergence; if the convergence is linear give the rate.

a. $x_{n+1} = \frac{6}{x_n - 1}$ $\qquad \alpha = 3$

b. $x_{n+1} = \frac{2x_n}{3} + \frac{1}{3x_n^2}$ $\qquad \alpha = 1$

c. $x_{n+1} = \log(1 + x_n) - (x_n - \frac{1}{2}x_n^2)$ $\qquad \alpha = 0$

Answer:

Put each in the form $x_{n+1} = g(x_n)$ and look at derivatives of $g(x)$.

a.

$$\begin{aligned} g(x) &= \frac{6}{x-1} \\ g'(x) &= \frac{-6}{(x-1)^2} \\ g'(3) &= -\frac{3}{2} \qquad |g'(\alpha)| > 1.0 \Rightarrow \text{Iteration diverges} \end{aligned}$$

b.

$$\begin{aligned} g(x) &= \frac{2x}{3} + \frac{1}{3x^2} \\ g'(x) &= \frac{2}{3} - \frac{2}{3x^3} \\ g'(1) &= 0 \\ g''(x) &= \frac{6}{3x^2} = \frac{2}{x^4} \\ g''(1) &= 2 \neq 0 \qquad \Rightarrow \text{Converges with order} = 2. \end{aligned}$$

c.

$$\begin{aligned} g(x) &= \log(1+x) - (x - \frac{x^2}{2}) \\ g'(x) &= \frac{1}{1+x} - 1 + x \\ g'(0) &= 0 \\ g''(x) &= \frac{-1}{(1+x)^2} + 1 \\ g''(0) &= 0 \\ g'''(x) &= \frac{2}{\quad} \end{aligned}$$