APPM 5600 NUMERICAL ANALYSIS

FINAL EXAM

TIME: 120 MINUTES December 12, 1994, 3:30–5:30 p.m.

Answer 8 of the 9 questions. Indicate your choice by striking out one number in the grade box below. No aids except calculators permitted

NAME:__________________________

For Grader Only	
1	/ 20
2	/ 20
3	/ 20
4	/ 20
5	/ 20
6	/ 20
7	/ 20
8	/ 20
9	/ 20
Σ	/160

1. Determine the values of a, b, c so that the following is a cubic spline with knots at 0,1,2:

$$s(x) = \begin{cases} 3 + x - 9x^2 & x \in [0,1] \\ a + b(x-1) + c(x-1)^2 + d(x-1)^3 & x \in [1,2] \end{cases}$$

Determine d that minimizes $\int_0^2 (s''(x))^2 dx$. Is there a value for d that makes $s(x)$ a natural spline? Explain your answer.

2. Consider the **normalized** orthogonal basis with respect to the weight function $w(x) = 1$ on the interval $[-1, 1]$ given by

$$\left\{\varphi_0(x) = \sqrt{\frac{1}{2}}, \quad \varphi_1(x) = \left(\sqrt{\frac{3}{2}}\right) x, \quad \varphi_2(x) = \frac{1}{2}\sqrt{\frac{5}{2}}\left(3x^2 - 1\right)\right\}.$$

   (a) Express $f(x) = x^2$ as a linear combination of the φ's. That is, write $f(x) = a\varphi_0(x) + b\varphi_1(x) + c\varphi_2(x)$ for some constants a, b, and c.

   (b) Find the polynomial of degree at most 1 that is closest to $f(x)$ on $[-1, 1]$ in the norm given by

$$||g||_2^2 = \int_{-1}^{1} g^2(x)dx.$$

   Hint: This is very easy if you've done (a) correctly.

3. Suppose we want to find $p(x) = ax + b$ that is closest to $f(x) = x^2$ on $[0, 1]$ with respect to the infinity norm:

$$||g||_\infty = \max_{0 \le x \le 1} |g(x)|.$$

   (a) Using the Chebyshev Equioscillation Theorem, show that $b \neq 0$.

   (b) Draw a picture of the error function $e(x) = f(x) - p(x)$. Where must the maxima and minima of $e(x)$ occur in the interval $[0, 1]$? How many of each must there be?

   (c) Using the picture as a guide, compute $p(x)$.

4. Let $x_i = x_0 + ih$, for $i = 1, \dots, n$ be a mesh of equally spaced points. Let $f(x) \in C^{(n+2)}[x_0, x_n]$ and let $p_n(x)$ be the unique polynomial of degree n that interpolates $f(x)$ on $x_0, \dots, x_n$.

   (a) Give an expression for the interpolation error, $\mathcal{E}_n = f(x) - p_n(x)$, in terms of divided differences of $f(x)$ and of derivatives of $f(x)$.

   (b) Give an expression for the error in the Newton-Cotes formula, $E_n(F) = \int_{x_0}^{x_n} f(x) - p_n(x)dx$, in terms of divided differences of $f(x)$ and of derivatives of $f(x)$.

   (c) What is the degree of precision of the Newton-Cotes formula?

   (d) Consider using a composite Simpson's rule to integrate $f(x) = e^x$ on the interval $[0, 1]$. Each time the mesh spacing h is halved, how would you expect the error to be affected? If the function were $f(x) = \sqrt{x}$ what would you expect? Explain your answer.

5. Given the matrix

$$A = \begin{bmatrix} 1 & -6 \\ 1 & 6 \\ 0 & 9 \\ -1 & 6 \\ 1 & 6 \end{bmatrix},$$

construct the Householder reflection, H, that reflects the first column of A onto the first coordinate axis. Then, compute HA. What are the singular values of A?

6. Let
$$A = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

   (a) What is the characteristic polynomial of A?

   (b) What is the minimal polynomial of A?

   (c) Consider the iteration:

$$\begin{aligned} (A - 5I)\underline{z}_n &= \underline{y}_{n-1} \\ \underline{y}_n &= \frac{\underline{z}_n}{\|\underline{z}_n\|} \\ \lambda_n &= \frac{\langle A\underline{y}_n, \underline{y}_n \rangle}{\langle \underline{y}_n, \underline{y}_n \rangle}. \end{aligned}$$

   Will the sequence $\{\lambda_n\}$ converge and, if so, to what?

7. Show that if

$$\int_a^b f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

is exact for all polynomials of degree at most $2n-1$, then the polynomial

$$p(x) = \prod_{i=1}^{n}(x - x_i)$$

is orthogonal to P_{n-1}, the set of all polynomials on $[a, b]$ of degree at most $n - 1$, with respect to the inner product

$$\langle f,\, g\rangle = \int_a^b f(x)g(x)dx.$$

That is, for every $g \in P_{n-1}$, show that $\langle p,\, g\rangle = 0$.

8. Let $f(x)$ be three times continuously differentiable near a root α. Define

$$u(x) = \frac{f(x)}{f'(x)}$$

and apply Newton's method to $u(x)$. Show the resulting iteration will converge quadratically to α regardless of the multiplicity of the root α, provided x_0 is sufficiently close to α. (Hint: write $f(x) = (x - \alpha)^p h(x)$ where $h(\alpha) \neq 0$.)

9. Which of the following iterations will converge to the indicated fixed point(provided x_0 is sufficiently close to α). If it does converge, give the order of convergence; if the convergence is linear give the rate.

   a. $x_{n+1} = \frac{20}{1+x_n}$ $\qquad \alpha = 4$

   b. $x_{n+1} = \frac{2}{3}x_n + 9\frac{1}{x_n^2}$ $\qquad \alpha = 3$

   c. $x_{n+1} = e^{x_n} - (1 + 2x_n + \frac{1}{2}x_n^2)$ $\qquad \alpha = 0$