APPM 5600 NUMERICAL ANALYSIS

TEST I

TIME: 50 MINUTES October 6, 2004, 10:00–10:50 a.m.

No aids except calculators permitted

NAME:__________________________

For Grader Only	
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1. Given the mesh $x_0 = -2.0$, $x_1 = -1.0$, $x_2 = 1.0$, $x_3 = 2.0$

   i) Construct the Lagrange interpolating polynomial basis. (Leave in factored form: $p(x) = C(x-a)(x-b)(x-c)$)

   ii) Find the cubic polynomial, $C(x)$, that interpolates $f(x) = \sin(\frac{\pi}{4}x)$ on the given mesh. Use the factored forms derived above to simplify.

   iii) What is $f[x_0, x_1, x_2, x_3]$? (No major calculation should be necessary)

Answer:

i) The Lagrange basis is

$$\begin{aligned}
\ell_0 &= \tfrac{(x+1)(x-1)(x-2)}{(-2+1)(-2-1)(-2-2)} &= -\tfrac{1}{12}(x+1)(x-1)(x-2) \\
\ell_1 &= \tfrac{(x+2)(x-1)(x-2)}{(-1+2)(-1-1)(-1-2)} &= \tfrac{1}{6}(x+2)(x-1)(x-2) \\
\ell_2 &= \tfrac{(x+2)(x+1)(x-2)}{(1+2)(1+1)(1-2)} &= -\tfrac{1}{6}(x+2)(x+1)(x-2) \\
\ell_3 &= \tfrac{(x+2)(x+1)(x-1)}{(2+2)(2+1)(2-1)} &= \tfrac{1}{12}(x+2)(x+1)(x-1)
\end{aligned}$$

ii)

$$\begin{aligned}
C(x) &= \frac{1}{12}\sin(\frac{\pi}{2})(x+1)(x-1)(x-2) - \frac{1}{6}\sin(\frac{\pi}{4})(x+2)(x-1)(x-2) \\
&- \frac{1}{6}\sin(\frac{\pi}{4})(x+2)(x+1)(x-2) + \frac{1}{12}\sin(\frac{\pi}{2})(x+2)(x+1)(x-1) \\
&= \frac{1}{6}x(x+1)(x-1) - \frac{1}{6}\sqrt{2}x(x+2)(x-2) \\
&= \frac{1}{6}x(x^2-1) - \frac{\sqrt{2}}{6}x(x^2-4) \\
&= -\frac{(\sqrt{2}-1)}{6}x^3 + \frac{(4\sqrt{2}-1)}{6}x
\end{aligned}$$

iii) The leading coefficient of $C(x)$ is $f[x_0, x_1, x_2, x_3]$:

$$f[x_0, x_1, x_2, x_3] = -\frac{(\sqrt{2}-1)}{6}$$

2. **i)** Construct the divided difference table for Hermite interpolation on the mesh $x_1 = 1.0, x_2 = 2.0$ using the values $f(x_1)$, $f'(x_1)$, $f(x_2)$, $f'(x_2)$

   ii) Construct the Hermite interpolating polynomial to the function $f(x) = \ln(x)$ using the divided difference table. (You do not need to simplify the polynomial.)

   iii) Write a **formula** for the error in the interpolating polynomial at the point t that involves derivatives of $f(x)$.

   iv) For $f(x) = \ln(x)$, derive a bound on the error for $t \in [1, 2]$. This should be a number.

Answer:

i) The divided difference table is

$$
\begin{array}{lllll}
1 & f(x_1) & & & \\
 & & f'(x_1) & & \\
1 & f(x_1) & & ((f(x_2) - f(x_1)) - f'(x_1)) & \\
 & & f(x_2) - f(x_1) & & (f'(x_2) - 2(f(x_2) - f(x_1)) + f'(x_1)) \\
2 & f(x_2) & & (f'(x_2) - (f(x_2) - f(x_1))) & \\
 & & f'(x_2) & & \\
2 & f(x_2) & & &
\end{array}
$$

ii) The Hermite interpolating polynomial is

$$
\begin{aligned}
p(x) &= f(x_1) + (x-1)f'(x_1) + (x-1)^2(f(x_2) - f(x_1) - f'(x_1)) \\
&+ (x-1)^2(x-2)(f'(x_2) - 2(f(x_2) - f(x_1)) + f'(x_1))
\end{aligned}
$$

Substituting $f(x) = \ln(x)$ yields

$$p(x) = (x-1) + (x-1)^2(\ln(2) - 1) + (x-1)^2(x-2)(\frac{1}{2} - 2\ln(2) + 1)$$

iii) The error may be written as

$$e(t) = f(t) - p(t) = \frac{(x-1)^2(x-2)^2}{4!} f^{(4)}(\xi) \qquad \text{for some } \ t \in H(1, 2, t)$$

iv) If $f(x) = \ln(x)$, then $f^{(4)}(x) = \frac{-6}{x^4}$. This yields

$$
\begin{aligned}
\max_{t\in[1,2]} |e(t)| &\leq \frac{1}{24} \max_{t\in[1,2]} |(t-1)^2(t-2)^2| \max_{t\in[1,2]} \frac{6}{t^4} \\
&= \frac{1}{4}\frac{1}{16} = \frac{1}{64}
\end{aligned}
$$

3. For each of the following determine if the iteration will converge to the indicated root using any starting guess in the indicated interval. If the iteration will converge, determine the order of convergence. If the convergence is linear, determine the rate.

   i) $\alpha = 5$ $[0,\ 10]$ $x_{i+1} = \frac{1}{2}(x+5)$

   ii) $\alpha = 0$ $[0,\ 1/2]$ $x_{i+1} = e^{2x_i} - (1 + 2x_i + 2x_i^2)$

   iii) $\alpha = 0$ $[-1,\ 1]$ $x_{i+1} = \frac{2}{\pi}\sin(\frac{\pi}{2}x_i)$ (Hint: draw a picture.)

Answer:

i) Note that $g(x) = \frac{1}{2}(x+5)$ is monotonically increasing on $[0,\ 10]$ and that $g(0) = 2.5 > 0$ and $g(10) = 7.5 < 10$. This implies that $g([0,\ 10]) \subseteq [0,\ 10]$. Next,

$$g'(x) = 1/2 < 1.0$$

The iteration will converge linearly with rate 1/2.

ii) Note that $g(x) = e^{2x} - (1 - 2x + 2x^2)$ is increasing on $[0,\ 1/2]$ and that $g(1/2) = e^x - 2.5 \le 1/2$. Thus, $g([0,\ 1/2]) \subset [0,\ 1/2]$.

Next we see that

$$\begin{aligned} g'(x) &= 2e^{2x} - (2+4x), & g'(0) = 0 \\ g''(x) &= 4e^2x - 4, & g''(0) = 0 \\ g'''(x) &= 8e^2x & g'''(0) \neq 0 \end{aligned}$$

For x_0 sufficiently close to the solution, the iteration will converge to the unique solution $\alpha = 0$ with order 3. However, we must check to see that $|g'(x)| < 1.0$ for every $x \in [0,\ 1/2]$. Note that

$$g'(1/2) = 2e - 4 > 5.4 - 4 = 1.4$$

Thus, we cannot guarantee convergence for ever $x_0 \in [0,\ 1/2]$.

iii) Clearly, $\frac{2}{\pi}\sin(\frac{\pi}{2}[-1,1]) \subseteq [-1,\,1]$. We also see that

$$g'(x) = \cos(\frac{\pi}{2}x)$$

which yields

$$0 \leq g'(x) \leq 1$$

for every $x \in [-1,\,1]$. However, $g'(0) = 1.0$ so we cannot say $\max_{x\in[-1,\,1]} |g'(x)| < 1.0$ The answer is not totally clear from these facts. Note that for $x \neq 0$ we have

$$0 \leq g'(x) < 1$$

If $x_0 > 0$ the iterates will be monotonically decreasing. If $x_0 < 0$ the iterates will be monotonically increasing. The iteration will converge, but it will converge sublinearly; that is,

$$|\alpha - x_{i+1}| \leq \lambda_i|\alpha - x_i|$$

where $\lambda_i < 1$, but $\lambda_i \to 1.0$.

4. Consider the fixed-point iteration

$$x_{n+1} = \frac{1}{6}(1 - e^{(x_n + y_n)}),$$

$$y_{n+1} = \frac{1}{3}((x_n - y_n) - \frac{2}{\pi}\sin(\frac{\pi}{2}(x_n - y_n)),$$

i) Determine if the fixed point iteration will converge to the fixed-point $(x, y) = (0,\, 0)$ if the initial guess $(x_0,\, y_0)$ is sufficiently close to $(0,\, 0)$.

ii) Determine the rate of linear convergence of the method in the max norm.

iii) Will the iteration converge for any initial guess in the set

$$D = \{(x, y) \;\; : \;\; -1 < (x+y) < 1, \quad -1 < (x-y) < 1\}$$

First write two criteria that need to be established. Then determine if the criteria are satisfied. (Use: $e \approx 2.72 < 3.0$)

Answer: This iteration is based on the equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underline{g}(x, y) = \begin{pmatrix} g_1(x, y) \\ g_2(x, y) \end{pmatrix} = \begin{pmatrix} \frac{1}{6}(1 - e^{(x+y)}) \\ \frac{1}{3}((x-y) - \frac{2}{\pi}\sin(\frac{\pi}{2}(x-y)) \end{pmatrix}.$$

Define

$$G(x, y) = \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{bmatrix} = \begin{bmatrix} -\frac{1}{6}e^{(x+y)} & -\frac{1}{6}e^{(x+y)} \\ \frac{1}{3}(1 - \cos(\frac{\pi}{2}(x-y))) & -\frac{1}{3}(1 - \cos(\frac{\pi}{2}(x-y))) \end{bmatrix}$$

Thus,

$$G(0, 0) = \begin{bmatrix} -\frac{1}{6} & -\frac{1}{6} \\ 0 & 0 \end{bmatrix}$$

which yileds

$$\|G(0, 0)\|_\infty = \frac{1}{3} < 1.0.$$

i) The iteration will converge for (x_0, y_0) sufficiently close to $(0,\, 0)$

ii) The rate of linear convergence will be bounded by $\lambda = 1/3$.

iii) We must show

a. $g(D) \subseteq D$

b. $\|G(x,y)\|_\infty < 1.0$ for every $(x,y) \in D$

To show **a.**, let $z = x + y$ and note that for $|z| \leq 1.0$ we have

$$|g_1(x,y)| \leq \frac{1}{6}|e-1| < \frac{1}{3}.$$

Let $w = x - y$ and note that for $|w| \leq 1.0$ we have

$$|g_2(x,y)| \leq \frac{1}{3}|w| \leq \frac{1}{3}.$$

Thus

$$\begin{aligned} |g_1(x,y) + g_2(x,y)| &\leq \frac{2}{3} \\ |g_1(x,y) - g_2(x,y)| &\leq \frac{2}{3}, \end{aligned}$$

which established $g(D) \subseteq D$.

To show **b.**, we see that for $(x,y) \in D$ we have

$$|G(x,y)| = \begin{bmatrix} |\frac{\partial g_1}{\partial x}| & |\frac{\partial g_1}{\partial y}| \\ |\frac{\partial g_2}{\partial x}| & |\frac{\partial g_2}{\partial y}| \end{bmatrix} \leq \begin{bmatrix} \frac{1}{6}e & \frac{1}{6}e \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix},$$

which yileds

$$\|G(x,y)\|_\infty \leq \max[\frac{e}{3}, \frac{2}{3}] = \frac{e}{3} < 1.0$$

Thus, the iteration will converge for any initial guess $(x_0,\, y_0) \in D$.