

APPM 5600

NUMERICAL ANALYSIS

TEST II

TIME: 50 MINUTES

November 5, 2004, 10:00–10:50 a.m.

No aids except calculators permitted

NAME: _____

For Grader Only	
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2	/ 25
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1. a.) State the Theorem of de la Vallee-Poussin

b.) Let $f(x) \in C[0, 2]$ and let

$$\rho_n(f) := \min_{q_n \in \mathcal{P}_n} \max_{x \in [0, 2]} |f(x) - q_n(x)|,$$

where $\mathcal{P}_n = \{ \text{polynomials of degree } \leq n \}$. Let $q_4(x) \in \mathcal{P}_4$ be such that the extreme values of $f(x) - q_4(x)$ on $[0, 2]$ are:

x_i	0.0	.200	.710	1.33	1.82	2.00
$e(x_i)$	$-1.36e - 3$	$1.38e - 3$	$-1.43e - 3$	$1.51e - 3$	$-1.57e - 3$	$1.60e - 3$

(Assume these are the only local extrema in $[0, 2]$.) Give upper and lower bounds on $\rho_4(f)$. Justify your answer.

Answer:

a.) See Theorem 4.9 in Atkinson, page 222

b.) By the Theorem of de la Vallee-Poussin, we know that

$$\min_j |f(x_j) - q_3(x_j)| \leq \rho_3(f) \leq \|f - q_3\|_\infty$$

The problem statement implies that

$$\|f - q_3\|_\infty = \max_j |f(x_j) - q_3(x_j)|$$

This yields the bounds

$$1.36e - 3 \leq \rho_3(f) \leq 1.60e - 3$$

2. a.) Find the **monic** orthogonal polynomials $\{p_j(x)\}_{j=0}^3$, with respect to the weight function $W(x) = (1 - x^2)$ on the interval $[-1, 1]$.
- b.) Orthogonal polynomials satisfy a three-term recursion of the type

$$p_{n+1}(x) = (a_n x + b_n)p_n(x) - c_n p_{n-1}(x)$$

Determine the exact values of the coefficients a_n , b_n and a formula for c_n in the recursion for the monic orthogonal polynomials from part a.). Prove that $c_n > 0$.

(Hint: Use symmetry to simplify the calculations.)

Answer:

- a.) Since the interval and the weight function are symmetric about the origin, we know that the orthogonal polynomials of even degree will be even and of odd degree will be odd. Let

$$\langle f, g \rangle_W := \int_{-1}^1 f(x)g(x)(1 - x^2)dx$$

The monic orthogonal polynomials are:

$$\begin{aligned} p_0(x) &= 1 \\ p_1(x) &= x \\ p_2(x) &= x^2 - \alpha & \alpha &= \frac{\langle x^2, 1 \rangle_W}{\langle 1, 1 \rangle_W} = \frac{\frac{4}{15}}{\frac{4}{3}} = \frac{1}{5} \\ p_3(x) &= x^3 - \beta x & \beta &= \frac{\langle x^3, x \rangle_W}{\langle x, x \rangle_W} = \frac{\frac{4}{35}}{\frac{4}{15}} = \frac{3}{7} \end{aligned}$$

- b.) Orthogonal polynomials satisfy a three-term recursion of the type

$$p_{n+1}(x) = (a_n x + b_n)p_n(x) - c_n p_{n-1}(x)$$

The monic orthogonal polynomials require $a_n = 1$. By symmetry, we know that if n is odd then $p_n(x)$ is an odd polynomial, and if n is even then $p_n(x)$ is an even polynomial. This implies that $b_n = 0$. Finally, we have the formula:

$$c_n = \frac{\langle x p_n, p_{n-1} \rangle_W}{\langle p_{n-1}, p_{n-1} \rangle_W} = \frac{\langle p_n, x p_{n-1} \rangle_W}{\langle p_{n-1}, p_{n-1} \rangle_W}$$

Using the recursion, we see

$$x p_{n-1}(x) = p_n(x) + c_{n-1} p_{n-2}(x)$$

Plugging this expression in above yields

$$c_n = \frac{\langle p_n, p_n \rangle_W}{\langle p_{n-1}, p_{n-1} \rangle_W} + c_{n-1} \frac{\langle p_n, p_{n-1} \rangle_W}{\langle p_{n-1}, p_{n-1} \rangle_W} = \frac{\langle p_n, p_n \rangle_W}{\langle p_{n-1}, p_{n-1} \rangle_W} > 0$$

3. Find the Gauss quadrature formula for $n = 3$ on the interval $[-1, 1]$ with weight function $W(x) = (1 - x^2)$. That is, find the weights and nodes for the formula

$$\int_{-1}^1 f(x)(1 - x^2)dx = \sum_{i=1}^3 \omega_i f(x_i) + E_3(f).$$

What is the degree of precision of this formula? (Hint: use the answer from problem 2. and symmetry to simplify the problem.)

Answer:

The quadrature nodes are the roots of $p_3(x)$.

$$x_1 = -\sqrt{\frac{3}{7}}, \quad x_2 = 0, \quad x_3 = \sqrt{\frac{3}{7}},$$

The weights are:

$$\omega_1 = \frac{14}{45}, \quad \omega_2 = \frac{32}{45}, \quad \omega_3 = \frac{14}{45}.$$

The degree of precision of the formula is 5

Details: We will show two ways to compute the weights. The weights satisfy

$$\omega_i = \int_{-1}^1 \ell_i(x)W(x)dx, \quad i = 1, 2, 3.$$

Thus,

$$\begin{aligned} \omega_1 &= \int_{-1}^1 \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}(1 - x^2)dx = \frac{7}{6} \int_{-1}^1 (x^2 - \sqrt{\frac{3}{7}}x)(1 - x^2)dx \\ &= \frac{7}{6} \int_{-1}^1 x^2(1 - x^2)dx = \frac{7}{6} \frac{4}{15} = \frac{14}{45}. \end{aligned}$$

By symmetry $\omega_3 = \omega_1$. Finally,

$$\begin{aligned} \omega_2 &= \int_{-1}^1 \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}(1 - x^2)dx = -\frac{7}{3} \int_{-1}^1 (x^2 - \frac{3}{7})(1 - x^2)dx \\ &= -\frac{7}{3} \left(\frac{4}{15} - \frac{3}{7} \frac{4}{3} \right) = \frac{32}{45}. \end{aligned}$$

(see next page)

There is a straightforward way to establish the weights. The formula must be exact for all polynomials of degree less than or equal to 5. Letting $f(x) = 1$, we have

$$\int_{-1}^1 (1 - x^2) dx = \frac{4}{3} = \omega_1 + \omega_2 + \omega_3$$

Letting $f(x) = x$, we have

$$\int_{-1}^1 x(1 - x^2) dx = 0 = \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 = -\sqrt{\frac{3}{7}}\omega_1 + \sqrt{\frac{3}{7}}\omega_3$$

Letting $f(x) = x^2$, we have

$$\int_{-1}^1 x^2(1 - x^2) dx = \frac{4}{15} = \omega_1 x_1^2 + \omega_2 x_2^2 + \omega_3 x_3^2 = \frac{3}{7}\omega_1 + \frac{3}{7}\omega_3$$

These three equations yield the answer above.

Finally, both the nodes and the roots could have been found from a direct calculation assuming symmetry ($x_1 = -x_3$, $x_2 = 0$ and $\omega_1 = \omega_3$) and using the additional equation

$$\int_{-1}^1 x^2(1 - x^2) dx = \frac{4}{35} = \omega_1 x_1^4 + \omega_2 x_2^4 + \omega_3 x_3^4$$

4. **a.** Given a mesh of equally spaced nodes $a = x_0 < x_1 < \dots < x_n = b$, write the composite Simpson's rule for approximating the integral $I(f) := \int_a^b f(x)dx$ together with an expression of the error, $E_n(f)$, in terms of the derivatives of f .
- b.** How many function evaluations would be required to guarantee an approximation of $\int_1^2 \frac{1}{x} dx$ to within a tolerance of $tol = \frac{2}{15} \times 10^{-4}$
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Answer:

- a.** The composite Simpson's rule can be written as

$$I_n(f) = \frac{h}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)].$$

where $h = (b - a)/n$. The error can be written

$$E_n(f) = I(f) - I_n(f) = -\frac{h^4(b-a)}{180} f^{(4)}(\eta), \quad \eta \in [a, b]$$

(Remark: The asymptotic form of the error,

$$\tilde{E}_n(f) = -\frac{h^4}{180} [f^{(4)}(b) - f^{(4)}(a)].$$

could be used. This results in a slightly different answer and does not provide a tight bound on the error.

- b.** With $f(x) = \frac{1}{x}$ we have

$$\max_{x \in [1, 2]} |f^{(4)}(x)| := \|f^{(4)}\|_\infty = \left\| \frac{24}{x^5} \right\|_\infty = 24$$

We seek $h = 1/n$ such that

$$\frac{h^4(b-a)}{180} \|f^{(4)}\|_\infty = \frac{h^4}{180} 24 = h^4 \frac{2}{15} \leq tol = \frac{2}{15} \times 10^{-4}$$

This yields

$$h^4 \leq 10^{-4}$$

or

$$h \leq .1$$

which is accomplished with $n = 10$. **The final answer is $n + 1 = 11$ functional evaluations.**