APPM 5600 **Numerical Analysis**

Test II, Take Home **Due: November 14, 1994**

Please answer the following questions briefly but completely. Do your own work. You may share references, but answer the questions by yourself.

Unless otherwise stated, assume $\underline{x}$, $\underline{y} \in \mathcal{C}^n$ and define the inner product

$$\langle \underline{x},\, \underline{y} \rangle = \sum_{i=1}^{n} x_i \overline{y}_i = \overline{\langle \underline{y},\, \underline{x} \rangle}.$$

Let $\|\underline{x}\|$ (without subscript) denote the induced norm and let $\|A\|$ denote the subordinate matrix norm.

1. Assume that A is an $n \times n$ real matrix. For parts i) - iii) assume A is symmetric.

   i) Show that if $\lambda \in \Sigma(A)$ (is an eigenvalue of A), then $\lambda \in \Re$ (is real).

   ii) Show that if $\underline{x} \in \mathcal{C}^n$, then $\langle A\underline{x},\, \underline{x} \rangle \in \Re$.

   iii) Let

   $$\mu_1 = \min_{\underline{x} \in \mathcal{C}^n, \underline{x} \neq 0} \frac{\langle A\underline{x},\, \underline{x} \rangle}{\langle \underline{x},\, \underline{x} \rangle}, \qquad \mu_2 = \max_{\underline{x} \in \mathcal{C}^n, \underline{x} \neq 0} \frac{\langle A\underline{x},\, \underline{x} \rangle}{\langle \underline{x},\, \underline{x} \rangle}.$$

   Show that $\Sigma(A) \subseteq [\mu_1,\, \mu_2]$.

   For iv) and v), assume that A is skewsymmetric ($A^t = -A$) and real.

   iv) Show that if $\underline{x} \in \mathcal{C}^n$ then $\langle A\underline{x},\, \underline{x} \rangle = i\alpha$ for some $\alpha \in \Re$.

   v) Let $\eta = \|A\|$. Show that if $\lambda \in \Sigma(A)$, then $\lambda = i\alpha$ for some $\alpha \in \Re$ with $|\alpha| \leq \eta$.

   Now assume that A is a general $n \times n$, real matrix. Let

   $$A_S = \frac{1}{2}(A + A^t), \qquad A_N = \frac{1}{2}(A - A^t)$$

   be the symmetric and skewsymmetric parts of A.

   vi) Use what you have proved above to construct a region in the complex plane that includes $\Sigma(A)$.

2. Consider the least-squares problem

$$A\underline{x} \simeq \underline{b}.$$

where A is an $m \times n$ real matrix with $m > n$.

i) Briefly describe the QR factorization method for finding the least squares solution. Assume $rank(A) = n$.

ii) Assume that column pivoting is used in the QR factorization. Describe what happens if $rank(A) = p < n$.

iii) Assume that $rank(A) = p < n$ and that a QR factorization has been performed. Describe an algorithm for finding the shortest least squares solution.

3. Let $p_k(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_k x^k$ be a polynomial of degree k. Briefly describe the companion matrix method for finding all of the roots of $p_k(x)$. Explain why the QR method for finding eigenvalues is especially well suited for finding the eigenvalues of the comapnion matrix.

4. Let A be an $n \times n$ matrix. Define the Frobenius norm of A:

$$\|A\|_F^2 = \sum_{\substack{i\,=\,1,\ldots,n \\ j\,=\,1,\ldots,n}} |a_{i,j}|^2.$$

i) Show that if U is a unitary matrix, then

$$\|UA\|_F = \|A\|_F.$$

ii) Consider the singular value decomposition of A:

$$A = U\Sigma V^*.$$

Show that

$$\|A\|_F^2 = \sum_{i=1}^{n} \sigma_i^2 \geq \|A\|^2.$$

iii) Assume that A is nonsingular and consider the perturbation E. What is the perturbation E of smallest Frobenius norm such that $A + E$ is singular? Find a bound on $\|E\|_F$ that would guarantee that $A + E$ is nonsingular. (Hint: consider $A + E = U\Sigma V^* + E$.)