APPM 5600 NUMERICAL ANALYSIS

TEST II

TIME: 50 MINUTES November 11, 1996, 11:00–11:50 a.m.

No aids except calculators permitted

NAME:__________________________

For Grader Only

1	/ 10
2	/ 30
3	/ 30
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1. Write the modified Gram-Schmidt Algorithm for constructing $\{\phi_n(x)\}$, the sequence of orthogonal polynomials with respect to the inner product $(f,\, g) = \int_a^b f(x)g(x)w(x)dx$. What simplification occurrs naturally?

Answer:

$$\begin{aligned}
\phi_0(x) &= 1, \\
\phi_1(x) &= x - \sigma_{0,0}\phi_0(x), \\
&\vdots \\
\phi_{k+1}(x) &= x\phi_k(x) - \sigma_{k,\,k}\phi_k(x) - \sigma_{k,k-1}\phi_{k-1}(x),
\end{aligned}$$

where

$$\sigma_{k,j} = \frac{(x\phi_k,\, \phi_j)}{(\phi_j,\, \phi_j)}.$$

The recursion requires only three terms because

$$\sigma_{k,j} = 0$$

for $j < k - 1$.

2. Let $f(x) = x - x^3$. Find $p_1(x)$, the minimax best polynomial approximation of degree 1 on the interval $[-1, 1]$. What is $p_2(x)$, the minimax best polynomial of degree 2?

   Hint: Sketch a graph of $f(x)$ and use symmetry to reduce the problem as much as possible. Find $p_1(x)$ first, then $p_2(x)$ should be obvious.

Answer:

$p_1(x) = \frac{1}{4}x$

$p_2(x) = p_1(x)$

Proof: By symmetry, $p_1(x)$ must have a root at the origin, that is, $p_1(x) = ax$ for some a. Thus, we may confine our attention to the interval $[0, 1]$. Define

$$e(x) := x - x^3 - ax.$$

Appealing to the Chebyshev Equioscillation Theorem (CET),

$$\max_{x \in [0,\, 1]} e(x) = -e(1) = a.$$

Let $\hat{x} := arg\max_{x \in [0,\, 1]} e(x)$. The above equation becomes

$$\hat{x} - \hat{x}^3 - a\hat{x} = a,$$

which yields

$$a = \hat{x}(\hat{x} - 1).$$

We also know that

$$e'(\hat{x}) = 1 - 3\hat{x}^2 - a = 0,$$

which yields

$$a = 1 - 3\hat{x}^2.$$

Eliminating a in the two equations above yields

$$\hat{x}(\hat{x} - 1) = 1 - 3\hat{x}^2 \quad \Rightarrow \quad 2\hat{x}^2 + \hat{x} - 1 = 0,$$

or $\hat{x} = \frac{1}{2}$. This, in turn, yields $a = \frac{1}{4}$.

Finally, notice that $p_1(x)$ satisfies the CET for degree 2. Thus, $p_2(x) = p_1(x)$.

3. Find the Gauss quadrature formula

$$G_3(f) = \sum_{i=1}^{3} \omega_i f(x_i),$$

on the interval $[-1, 1]$ with respect to the weight function $w(x) = \cos(\frac{\pi}{2}x)$. What is the degree of precision of this formula?

Use the following definitions:

$$A := \int_{-1}^{1} \cos(\frac{\pi}{2}x)dx, \qquad B := \int_{-1}^{1} x^2\cos(\frac{\pi}{2}x)dx, \qquad C := \int_{-1}^{1} x^4\cos(\frac{\pi}{2}x)dx.$$

Hint: Use symmetry to simplify your computations.

Answer: The nodes and weights are:

$$x_1 = -\sqrt{\tfrac{C}{B}} \qquad \omega_1 = \tfrac{B^2}{2C}$$

$$x_2 = \ 0 \qquad \omega_2 = A - \tfrac{B^2}{C}$$

$$x_3 = \ \sqrt{\tfrac{C}{B}} \qquad \omega_3 = \tfrac{B^2}{2C}$$

The degree of precision of $G_3(f)$ is: 5.

Proof: The nodes will be the roots the orthogonal polynomials of degree 3 with respect to the inner product

$$(f,\, g)_w := \int_{-1}^{1} f(x)g(x)\cos(\frac{\pi}{2}x)dx$$

By symmetry we know that the odd degree orthogonal polynomials are odd and the even degree orthogonal polynomials are even. Thus,

$$p_0(x) \ = \ 1$$

$$p_1(x) \ = \ x$$

$$p_2(x) \ = \ x^2 - \alpha$$

$$p_3(x) \ = \ = x^3 - \beta x$$

Using the orthogonality conditions yields

$$\alpha = \frac{(x^2,\,1)_w}{(1,\,1)_w} = \frac{B}{A} \qquad \beta = \frac{(x^3,\,x)_w}{(x,\,x)_w} = \frac{C}{B}$$

Thus,

$$x_1 = -\sqrt{C/B}, \qquad x_2 = 0, \qquad x_3 = \sqrt{C/B}.$$

To find the weights we appeal to the fact that $G_3(f)$ is exact for polynomials of degree 5 or less. We have the equations:

$$\begin{array}{cccccccc}
\omega_1 & + & \omega_2 & + & \omega_3 & = & \int_{-1}^{1}\cos(\frac{\pi}{2}x)dx & = A, \\
\omega_1 x_1 & + & & + & \omega_3 x_3 & = & \int_{-1}^{1} x\cos(\frac{\pi}{2}x)dx & = 0, \\
\omega_1 x_1^2 & + & & + & \omega_3 x_3^2 & = & \int_{-1}^{1} x^2\cos(\frac{\pi}{2}x)dx & = B.
\end{array}$$

The last two equations yield

$$\omega_1 = \omega_3 = \frac{B^2}{2C},$$

while the first yields

$$\omega_2 = A - 2\omega_1 = A - \frac{B^2}{C}.$$

4. We have studied the Newton-Coats formulas for $n > 0$, for example, the trapezoidal rule ($n = 1$) and Simpson's rule ($n = 2$). Now, consider the midpoint rule ($n = 0$). Write

$$I(f) := \int_a^b f(x)dx = I_0(f) + E_0(f)$$

where

$$I_0(f) = (b-a)f(\frac{a+b}{2})$$

Derive a bound for $E_0(f)$.

Hint: Let $c = (a+b)/2$, write $E_0(f)$ as the integral of the interpolation error, and use the "$w(x)$ trick" with

$$w(x) = \int_a^x (t-c)dt$$

Answer: Let $h = (b-a)$, then (find C, p, q)

$E_0(F) = Ch^p f^{(q)}(\eta) = \frac{h^3}{24} f''(\eta)$ some $\eta \in [a,\, b]$

Proof: Written in the form of interpolation error we have

$$f(x) - f(c) = (x-c)f[c,x].$$

Thus,

$$E_0(f) = \int_a^b (x-c)f[c,x]dx.$$

Define

$$w(x) := \int_a^x (t-c)dt, \quad \Rightarrow \quad w'(x) = (x-c).$$

We also have

$$w(a) = w(b) = 0, \quad \text{and} \quad w(x) \le 0 \text{ for } \ x \in [a,\, b].$$

Using integration by parts and the modified mean value theorem we have

$$\begin{aligned} E_0(f) &= \int_a^b w'(x)f[c,x]dx = w(x)f[c,x]|_a^b - \int_a^b w(x)f[c,x,x]dx \\ &= -f[c,\eta,\eta]\int_a^b w(x)dx = -f[c,\eta,\eta]\int_a^b\int_a^x (t-c)dtdx \qquad \eta \in [a,b] \\ &= -f[c,\eta,\eta]\int_a^b\int_a^x (t-c)dtdx = \frac{h^3}{24}f''(\xi) \qquad \xi \in [a,b] \end{aligned}$$