Box-Muller Transformation for Drawing N(0,1) Random Variables
Let U_1 and U_2 be independent $uniform(0,1)$ random variables. Then

$$\begin{aligned} X_1 &= \sqrt{-2\ln U_1}\cos(2\pi U_2) \\ X_1 &= \sqrt{-2\ln U_1}\sin(2\pi U_2) \end{aligned}$$

are **independent** $N(0,1)$ random variables!

This is a standard transformation problem. Let $f_{U_1,U_2}(u_1,u_2)$ be the joint pdf for U_1 and U_2 and let $x_1 = g_1(u_1,u_2)$ and $x_2 = g_2(u_1,u_2)$.

Then the joint pdf for X_1 and X_2 is given by

$$f_{X_1,X_2}(x_1,x_2) = f_{U_1,U_2}(g_1^{-1}(x_1,x_2), g_2^{-1}(x_1,x_2)) \cdot |J|$$

where J is the Jacobian of the transformation given by

$$J = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{bmatrix}$$

where $u_1 = g_1^{-1}(x_1,x_2)$ and $u_2 = g_2^{-1}(x_1,x_2)$.

Inverting the above transformations, we have

$$\begin{aligned} u_1 &= g_1^{-1}(x_1,x_2) &= exp\left(-\frac{x_1^2+x_2^2}{2}\right) \\ u_2 &= g_2^{-1}(x_1,x_2) &= \frac{1}{2\pi}tan^{-1}\left(\frac{x_2}{x_1}\right) \end{aligned}$$

Then the Jacobian is

$$J = \begin{bmatrix} -x_1\, exp\left(-\frac{x_1^2+x_2^2}{2}\right) & -x_2\, exp\left(-\frac{x_1^2+x_2^2}{2}\right) \\ \frac{-x_2/x_1^2}{2\pi(1+x_2^2/x_1^2)} & \frac{1/x_1}{2\pi(1+x_2^2/x_1^2)} \end{bmatrix}$$

The determinant is

$$|J| = \frac{-(1+x_2^2/x_1^2)\, exp\left(-\frac{x_1^2+x_2^2}{2}\right)}{2\pi(1+x_2^2/x_1^2)} = \frac{exp\left(-\frac{x_1^2+x_2^2}{2}\right)}{2\pi}$$

Finally, since the joint pdf for U_1 and U_2 is $f_{U_1,U_2}(u_1,u_2) = 1 \cdot I_{(0,1)}(u_1)I_{(0,1)}(u_2)$, we get

$$f_{X_1,X_2}(x_1,x_2) = \frac{exp\left(-\frac{x_1^2+x_2^2}{2}\right)}{2\pi} I_{(-\infty,\infty)}(x_1)I_{(-\infty,\infty)}(x_2)$$

which factors into two standard normal pdfs! Cool!