The Polar Transformation for Drawing N(0,1) Random Variables

Let U_1 and U_2 be independent $uniform(0, 1)$ random variables.

Let $V_1 = 2U_1 - 1$ and $V_2 = 2U_2 - 1$. (These are independent uniform(-1,1)'s.)

If $S := V_1^2 + V_2^2 \leq 1$, let

$$C = \sqrt{\frac{-2}{S}\ln(S)}.$$

Then $X_1 = CV_1$ and $X_2 = CV_2$ are indepdent $N(0, 1)$ random variables.

Proof:

Recall that from the Box-Muller Method, we know that

$$Y_1 := \sqrt{-2\ln(U_1)}\cos(2\pi U_2) \qquad \text{and} \qquad Y_2 := \sqrt{-2\ln(U_1)}\sin(2\pi U_2)$$

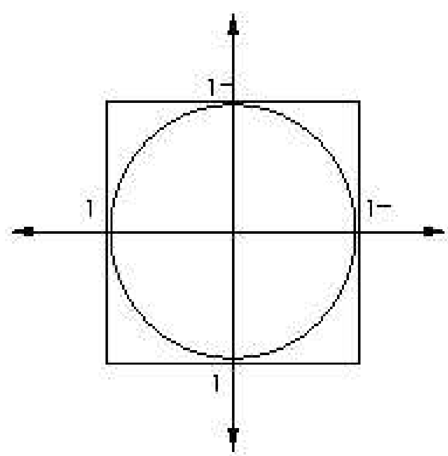
are independent and normally distributed $N(0, 1)$ random variables.

Let's write this as

$$Y_1 := \sqrt{-2\ln(U_1)}\cos(\theta) \qquad \text{and} \qquad Y_2 := \sqrt{-2\ln(U_1)}\sin(\theta)$$

where $\theta \sim unif(0, 2\pi)$.

Now $V_1 = 2U_1 - 1$ and $V_2 = 2U_2 - 1$ can be thought of as coordinates of a point that is uniformly distributed on the square shown below.



If this point (V_1, V_2) happens to fall in the unit circle (ie: $S := V_1^2 + V_2^2 \leq 1$), then it will also be uniformly distributed in the circle.

We assume that $S \leq 1$ (ie: (V_1, V_2) is uniformly distributed on the unit disk). (If not, we have to start over from the begining with new U_1 and U_2.)

We can transform V_1 and V_2 to polar coordinates and write:

$$V_1 = R\cos\Theta \qquad V_1 = R\sin\Theta$$

where R and Θ are random variables.

Let's find the joint density of R^2 and Θ.

(V_1, V_2) uniform on the unit disk implies that the joint pdf is

$$f_{V_1,V_2}(v_1, v_2) = \frac{1}{\pi} I_{(0,1)}(v_1^2 + v_2^2)$$

This is again a standard transformation problem.
Write (V_1, V_2) in polar coordinates:

Then

$$v_1 = g_1^{-1}(r^2, \theta) = r\,sin(\theta)$$

and

$$v_2 = g_2^{-1}(r^2, \theta) = r\,cos(\theta)$$

Notice that $R^2 = S := V_1^2 + V_2^2$

So

$$v_1 = g_1^{-1}(s, \theta) = \sqrt{s}\,sin(\theta)$$

and

$$v_2 = g_2^{-1}(s, \theta) = \sqrt{s}\,cos(\theta)$$

So

$$f_{S,\Theta}(s, \theta) = f_{V_1,V_2}(g_1^{-1}(s, \theta), g_2^{-1}(s, \theta)) \cdot |J|$$

where

$$J = \begin{vmatrix} \frac{\partial v_1}{\partial s} & \frac{\partial v_1}{\partial \theta} \\ \frac{\partial v_2}{\partial s} & \frac{\partial v_2}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{s}} sin(\theta) & \sqrt{s}\,cos(\theta) \\ \frac{1}{2\sqrt{s}}\,cos(\theta) & -\sqrt{s}\,sin(\theta) \end{vmatrix} = -\frac{sin^2(\theta)}{2} - \frac{cos^2(\theta)}{2}$$

So,

$$\begin{aligned} f_{S,\Theta}(s, \theta) &= f_{V_1,V_2}(\sqrt{s}\,sin(\theta), \sqrt{s}\,cos(\theta)) \cdot |-1| = -\tfrac{1}{2} \\ &= \tfrac{1}{\pi} I_{(0,1)}(s) I_{(0,2\pi)}(\theta). \end{aligned}$$

So we see that S and Θ are independent and are uniform on $(0, 1)$ and $(0, 2\pi)$, respectively.

We can now rewrite the Box-Muller transformations):

$$Y_1 := \sqrt{-2\ln(U_1)}\cos(\theta) = \sqrt{-2\ln(S)}\,\frac{V_2}{\sqrt{S}} = \sqrt{\frac{-2\ln(S)}{S}}V_2$$

and

$$Y_1 := \sqrt{-2\ln(U_1)}\sin(\theta) = \sqrt{-2\ln(S)}\,\frac{V_1}{\sqrt{S}} = \sqrt{\frac{-2\ln(S)}{S}}V_1$$