Worksheet 14 Solutions

① $x = t^3 \qquad y = 3t^2/2 \qquad 0 \le t \le \sqrt{3}$

a) $L = \int_0^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\sqrt{3}} \sqrt{(3t^2)^2 + (3t)^2}\, dt = \int_0^{\sqrt{3}} 3t\sqrt{t^2+1}\, dt$

$u = t^2+1 \qquad du = 2t\,dt \Rightarrow \qquad t\,dt = \frac{1}{2}du$

$L = \int_{u=1}^{4} \frac{3}{2} u^{1/2}\, du = u^{3/2}\Big|_{u=1}^{4} = 8 - 1 = \boxed{7}$

b) $SA = \int_0^{\sqrt{3}} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\sqrt{3}} \cancel{2}\pi \left(\frac{3t^2}{\cancel{2}}\right) 3t\sqrt{t^2+1}\, dt$

$= 9\pi \int_0^{\sqrt{3}} t^3 \sqrt{t^2+1}\, dt = 9\pi \int_0^{\sqrt{3}} \underbrace{t^2}_{u} \cdot \underbrace{t\sqrt{t^2+1}\, dt}_{dV}$ integrate by parts

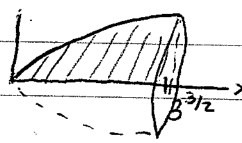
$du = 2t \qquad V = \frac{1}{3}(t^2+1)^{3/2}$

$= 9\pi \left[\frac{1}{3} t^2 (t^2+1)^{3/2} \Big|_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{2}{3} t (t^2+1)^{3/2}\, dt \right] = 9\pi \left[\frac{1}{\cancel{3}} \cdot \cancel{3} (2)^3 - \frac{2}{3} \cdot \frac{1}{5} (t^2+1)^{5/2} \right]_0^{\sqrt{3}}$

$= 9\pi \left[8 - \frac{2}{15}(2^5 - 1) \right] = 9\pi \left[8 - \frac{2^6}{15} + \frac{2}{15} \right]$

c) $\cancel{x} = t^3 \Rightarrow \quad t = x^{1/3} \qquad\qquad 0 \le t \le \sqrt{3}$

$y = \frac{3}{2} t^2 = \frac{3}{2} x^{2/3} = y \qquad \Rightarrow \quad 0 \le x \le 3^{3/2} \qquad 0 \le y \le \frac{9}{2}$



Disks:

$r = y = \frac{3}{2} x^{2/3}$

$V = \int_{x=0}^{3^{3/2}} \pi r^2\, dx = \int_0^{3^{3/2}} \frac{9\pi}{4} x^{4/3}\, dx$

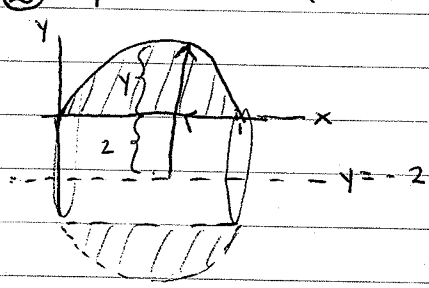
Shells:

$r = y$

$h = 3^{3/2} - x$

$V = \int_{y=0}^{9/2} 2\pi y \left[3^{3/2} - \left(\frac{2}{3}\right)^{3/2} y^{3/2} \right] dy$

② $y = x - x^2 = x(1-x) \qquad$ revolve about $y = -2$

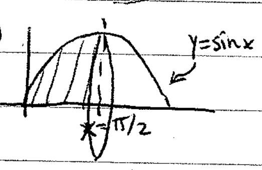


Disks:

$V = \int_{x=0}^{1} \pi r^2\, dx = \int_0^1 \pi (y+2)^2\, dx$

$= \int_0^1 \pi (x - x^2 + 2)^2\, dx$

③

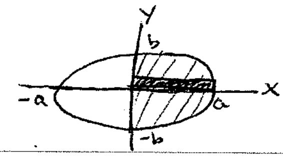


$r = y = \sin x$

$V = \int_0^{\pi/2} \pi \sin^2 x\, dx = \pi \int_0^{\pi/2} \frac{1 - \cos 2x}{2}\, dx$

$= \pi \left[\frac{x}{2} - \frac{1}{4} \sin 2x \right]_0^{\pi/2} = \pi \left(\frac{\pi}{4} \right) = \boxed{\frac{\pi^2}{4}}$

(4) CoM: inside $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$

$x = \sqrt{a^2 - \frac{a^2}{b^2}y^2}$

$\boxed{\bar{y} = 0 \text{ by symmetry}}$

horizontal strips:

$\tilde{x} = \frac{x}{2} \qquad \tilde{y} = y \qquad dm = \delta dA = x \cdot dy$

$\bar{x} = \frac{M_y}{M} = \frac{\int \tilde{x}\, dm}{\int dm}$

$M_y = \int \tilde{x}\, dm = \int_{-b}^{b} \frac{1}{2}\sqrt{a^2 - \frac{a^2}{b^2}y^2} \cdot \sqrt{a^2 - \frac{a^2}{b^2}y^2}\, dy$

$= \int_{-b}^{b} \frac{1}{2}\left(a^2 - \frac{a^2}{b^2}y^2\right) dy = \int_0^b a^2 - \frac{a^2}{b^2}y^2 dy = a^2 b - \frac{a^2}{3b^2}b^3$

$= a^2 b - \frac{1}{3}a^2 b = \frac{2}{3}a^2 b$

$M = \int_{-b}^{b} \sqrt{a^2 - \frac{a^2}{b^2}y^2}\, dy = \int_{-b}^{b} a\sqrt{1 - \frac{1}{b^2}y^2}\, dy = \int_{-\pi/2}^{\pi/2} ab\cos^2\theta\, d\theta$

$\frac{y^2}{b^2} = \sin^2\theta \qquad y = b\sin\theta \Rightarrow dy = b\cos\theta\, d\theta$

$= \int_{-\pi/2}^{\pi/2} ab\left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta = ab\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{-\pi/2}^{\pi/2} = ab\left[\frac{\pi}{2}\right] = \frac{\pi ab}{2}$

$\bar{x} = \frac{2}{3}\frac{a^2 b \cdot 2}{\pi ab} = \boxed{\frac{4a}{3\pi} = \bar{x}}$

(5) a) $\int \frac{1}{x^2+6x+25}\, dx = \int \frac{1}{(x+3)^2+16}\, dx = \int \frac{1}{16} \cdot \frac{1}{\left(\frac{x+3}{4}\right)^2+1}\, dx$

$u = \frac{x+3}{4} \qquad du = \frac{1}{4}dx$

$= \frac{4}{16}\int \frac{1}{u^2+1}\, du = \frac{1}{4}\arctan(u) + C = \boxed{\frac{1}{4}\arctan\left(\frac{x+3}{4}\right) + C}$

b) $\int \frac{1}{x^2+6x+8}\, dx = \int \frac{1}{(x+3)^2-1}\, dx = \int \frac{1}{[(x+3)+1][(x+3)-1]}\, dx$

$= \int \frac{1}{(x+4)(x+2)}\, dx = \int \frac{A}{x+4} + \frac{B}{x+2}\, dx$

$A(x+2) + B(x+4) = 1 \Rightarrow \begin{matrix} A+B=0 \\ 2A+4B=1 \end{matrix} \qquad B = 1/2 \qquad A = -1/2$

$= -\frac{1}{2}\ln|x+4| + \frac{1}{2}\ln|x+2| + C = \boxed{\frac{1}{2}\ln\left|\frac{x+2}{x+4}\right| + C}$

(6) a) $\int_1^\infty \frac{x^2+1}{x^5+x+1}\, dx \qquad \frac{x^2+1}{x^5+x+1} < \frac{x^2+1}{x^5} = \frac{1}{x^3} + \frac{1}{x^5}$

$\int_1^\infty \frac{1}{x^3} + \frac{1}{x^5}\, dx = -\frac{1}{2}x^{-2} - \frac{1}{4}x^{-4}\Big|_1^\infty$ $\boxed{\text{Converges}}$

b) $\int_0^2 \frac{1}{(1-x)^2}\,dx = \int_{u=1}^{-1} \frac{-du}{u^2} = \int_{-1}^{0} \frac{du}{u^2} + \int_0^1 \frac{du}{u^2} = \frac{-1}{u}\Big|_{-1}^{0} + \frac{-1}{u}\Big|_0^1$

$u = 1-x \quad du = -dx \qquad = -1 + \cancel{\lim_{u\to 0} \frac{-1}{u} + \lim_{u\to 0} \frac{1}{u}} - 1 = -2$

Converges

7 a) $\sum_{n=1}^{\infty} (-1)^n \ln(n)$ diverges by n^{th} term test

b) $\sum_{n=1}^{\infty} \frac{n^n}{3^n}$ N^{th} root test: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{n}{3} = \infty \Rightarrow$ diverges

c) $\sum_{n=1}^{\infty} \frac{3\ln(n)+2}{n^2+2n+1}$ $\quad \frac{3\ln(n)+2}{(n+1)^2} \leq \frac{3[\ln(n)+1]}{n^2}$

$\nwarrow$ converges by integral test using substitution $u = \ln x \quad du = \frac{1}{x}$
$x = e^u$

8 $\int_0^{1/10} e^{-x^2}\,dx = \int_0^{1/10} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\, n!}\Big|_{x=0}^{1/10} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(10)^{2n+1}\, n!\,(2n+1)}$

Error from keeping first 4 terms ($n=0$ to $n=3$)
$|\text{A.S. Error}| \leq |\ {}^{n=4}_{\text{term}}\ | = \frac{1}{10^9 \cdot 4! \cdot 9}$

9 $f(x) = \int_0^x \sin(t^2)\,dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{2(2n+1)}}{(2n+1)!}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+3}}{(4n+3)(2n+1)!}\Big|_0^x$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}$

$f(0.1)$ to accuracy of 4 decimal places $\Rightarrow 10^{-4}$

$f(0.1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(10)^{4n+3}(4n+3)(2n+1)!}$ $\quad |n=0 \text{ term}| = \frac{1}{10^3(3)} \not\leq 10^{-4}$

$|n=1 \text{ term}| = \frac{1}{10^7(7)(3!)} \leq 10^{-4}$ ✓

only need 1 term

10 a) $\sum_{n=1}^{\infty} \frac{n^5 (x-3)^{2n}}{4^n}$ Ratio Test: $\lim_{n\to\infty} \left| \frac{(n+1)^5 (x-3)^{2n+2}}{n^5 (x-3)^{2n}} \cdot \frac{4^n}{4^{n+1}} \right| = \frac{|x-3|^2}{4} < 1$

$\Rightarrow |x-3| < 2 \Rightarrow 1 < x < 5$

$x=1$ diverges by n^{th} term test
$x=5$ diverges by n^{th} term test

b) $\sum_{n=1}^{\infty} \frac{n(\ln x)^{n-1}}{x}$ Ratio Test: $\lim_{n\to\infty} \left| \frac{(n+1)(\ln x)^n}{x} \cdot \frac{x}{n(\ln x)^{n-1}} \right| = |\ln x| < 1$

$e^{-1} < x < e^1$ $\quad x = e^1$ and $x = e^{-1}$
diverge by n^{th} term test

c) Ratio Test: $\lim_{n\to\infty} \left| \frac{2^{n+1} x^{n+1}}{(3n+3)!\,(n+1)!} \cdot \frac{(3n)!\,n!}{2^n x^n} \right| = \lim_{n\to\infty} \left| \frac{2x}{(3n+3)(3n+2)(3n+1)(n+1)} \right| = 0 < 1$

Converges for all x

11 a) T b) T c) F d) T